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이학박사 학위논문

Arithmetic properties of the  
representations of ternary  
quadratic forms

(삼변수 이차형식 표현의 산술적 성질)

2016 년 8 월

서울대학교 대학원

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주 장 원

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(삼변수 이차형식 표현의 산술적 성질)

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# Arithmetic properties of the representations of ternary quadratic forms

A dissertation  
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of the requirements for the degree of  
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by

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# Abstract

In this thesis, we discuss some arithmetic relations on the representations of (positive definite integral) ternary quadratic forms. Let  $r(n, f)$  be the number of representations of an integer  $n$  by a ternary quadratic form  $f$  and let  $p$  be a prime such that  $f$  is isotropic over  $\mathbb{Z}_p$ . We show that under some restrictions,  $r(n, f)$  can be expressed as a summation of  $r(pn, g)$ 's and  $r(p^3n, g)$ 's with some extra term that can be explicitly computable, where each quadratic form  $g$  is contained in the same genus determined by  $f$  and  $p$ .

In the second part of the thesis, we discuss genus-correspondences between ternary quadratic forms respecting spinor genus. We modify the conjecture given by Jagy and prove this modified version. We also construct genus-correspondences satisfying some additional properties. In particular, we construct infinite family of genera of ternary quadratic forms that possess (absolutely) complete systems of spinor exceptional integers.

**Key words:** Representation of ternary quadratic forms, Watson transformations, Graph of ternary quadratic forms, Genus-correspondences, Complete system of spinor exceptional integers

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# Chapter 1

## Introduction

For a positive definite (non-classic) integral ternary quadratic form

$$f(x_1, x_2, \dots, x_k) = \sum_{1 \leq i \leq j \leq k} a_{ij} x_i x_j \quad (a_{ij} \in \mathbb{Z})$$

and an integer  $n$ , we say that  $n$  is represented by  $f$  if the diophantine equation  $f(x_1, x_2, \dots, x_k) = n$  has an integer solution  $(x_1, x_2, \dots, x_k) \in \mathbb{Z}^k$ . The set of all integers that are represented by  $f$  is denoted by  $Q(f)$ . It is quite an old and important problem to determine the set  $Q(f)$  explicitly. If the rank of  $f$  is two, then the set  $Q(f)$  is related with many other number theory subjects such as class field theory, and in fact, general theory of quadratic forms only gives some restricted information on the set  $Q(f)$ . If  $f$  is an indefinite form with rank greater than or equal to 3, then spinor genus theory gives an effective way to determine the set  $Q(f)$ . If  $f$  is a positive definite form with rank greater than 4, then one may compute an *effective* bound  $N$  such that every integer  $n$  greater than  $N$  is represented by  $f$  under the assumption that  $n$  is locally represented by  $f$ . Here the term “effectiveness” means that one may determine the set of integers less than  $N$  that are represented by  $f$  in a reasonable time. Recently, in [9], Hanke considered this problem when the rank is 4 and he computed an effective bound  $N$  such that every integer  $n$  greater than  $N$  that is primitively represented by  $f$  over  $\mathbb{Z}_p$ , for any prime  $p$  is also represented by itself. When  $f$  is a positive definite form with rank three, determining the set  $Q(f)$  is still remained unsolved.

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Let  $f$  be a positive definite (non-classic) integral ternary quadratic form  $f$ . For an integer  $n$ , we define

$$R(n, f) = \{(x_1, x_2, x_3) \in \mathbb{Z}^3 : f(x_1, x_2, x_3) = n\} \quad \text{and} \quad r(n, f) = |R(n, f)|.$$

It is well known that  $R(n, f)$  is always finite if  $f$  is positive definite. The theta series  $\theta_f(z)$  of  $f$  is defined by

$$\theta_f(z) = \sum_{n=0}^{\infty} r(n, f) e^{2\pi i n z},$$

which is in fact, a modular form of weight  $\frac{3}{2}$  and some character with respect to a certain congruence subgroup. Finding a closed formula for  $r(n, f)$  or finding all integers  $n$  such that  $r(n, f) \neq 0$  for an arbitrary ternary form  $f$  are quite old problems that are still widely open. As a simplest case, Gauss showed that if  $f$  is a sum of three squares, then  $r(n, f)$  is a multiple of the Hurwitz-Kronecker class number.

Though it seems to be quite difficult to find a closed formula for  $r(n, f)$ , some various relations between  $r(n, f)$ 's are known. One of the important relations comes from the Minkowski-Siegel formula. Let  $O(f)$  be the group of isometries of  $f$  and  $o(f) = |O(f)|$ . The weight  $w(f)$  of  $f$  is defined by  $w(f) = \sum_{[f'] \in \text{gen}(f)} \frac{1}{o(f')}$ , where  $[f']$  is the equivalence class containing  $f'$ . The Minkowski-Siegel formula says that the weighted sum of the representations by quadratic forms in the genus is, in principle, the product of local densities, that is,

$$\frac{1}{w(f)} \sum_{[f'] \in \text{gen}(f)} \frac{r(n, f')}{o(f')} = c^* \prod_p \alpha_p(n, f_p),$$

where the constant  $c^*$  can easily be computable and  $\alpha_p$  is the local density depending only on the local structure of  $f$  over  $\mathbb{Z}_p$ . Hence if the class number of  $f$  is one, then we have a closed formula on  $r(n, f)$ . As a natural modification of the Minkowski-Siegel formula, it was proved in [16] and [21] that the weighted sum of the representations of quadratic forms in the spinor genus is also equal to the product of local densities except spinor exceptional integers (see also [20] for spinor exceptional integers).

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Another important relation comes from the Watson transformation. Let  $p$  be a prime such that a unimodular component of  $f$  in a Jordan decomposition is anisotropic over  $\mathbb{Z}_p$ . Then one may easily show that

$$r(pn, f) = r(pn, \Lambda_p(f)),$$

where  $\Lambda_p(f)$  is defined in Section 2. Hence the theta series of  $f$  completely determines the theta series of  $\lambda_p(f)$ . Unfortunately if a unimodular component of the ternary form  $f$  over  $\mathbb{Z}_p$  is isotropic, one cannot expect such a nice relation. In this article, we consider the case when a unimodular component of the ternary form  $f$  over  $\mathbb{Z}_p$  is isotropic.

The subsequent discussion will be conducted in the more adapted geometric language of quadratic spaces and lattices. The term “lattice” will always refer to a positive definite non-classic integral  $\mathbb{Z}$ -lattice on an  $n$ -dimensional positive definite quadratic space over  $\mathbb{Q}$ . Here, a  $\mathbb{Z}$ -lattice is said to be *non-classic* if the norm ideal  $\mathfrak{n}(L)$  of  $L$  is contained in  $\mathbb{Z}$ . Let  $L = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \cdots + \mathbb{Z}x_n$  be a  $\mathbb{Z}$ -lattice of rank  $n$ . We write

$$L \simeq (B(x_i, x_j)).$$

The right hand side matrix is called a *matrix presentation* of  $L$ . We denote by  $\langle a, b, c, e, f, g \rangle$  for the ternary  $\mathbb{Z}$ -lattice with a matrix presentation

$$\begin{pmatrix} a & g & f \\ g & b & e \\ f & e & c \end{pmatrix},$$

for convenience. Furthermore for any integer  $a$ , we say that  $\frac{a}{2}$  is divisible by a prime  $p$  if  $p$  is odd and  $a \equiv 0 \pmod{p}$ , or  $p = 2$  and  $a \equiv 0 \pmod{4}$ . Any unexplained notations and terminologies can be found in [15] or [17].

In this thesis, we discuss some arithmetic relations on the representations of (positive definite integral) ternary quadratic forms. Parts of results in Chapters 3, 4 and 5 were proved in collaboration with Lee and Oh in [13], and most results in Chapter 6 were done by joint work with Oh in [14].

In Chapter 2, we introduce notations and terminologies that will be used

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in this thesis.

In Chapter 3, we collect some results related with so called, Watson's transformation. In particular, we focus on the behavior of the number of proper spinor genera under the Watson's transformation. We also generalize Watson's transformation in some direction and provide some formula on the weight sum of the representations of forms in a genus whose Watson's transformation is transformed to the same ternary form.

Let  $V$  be a (positive definite) ternary quadratic space and let  $L$  be a (non-classic) ternary  $\mathbb{Z}$ -lattice on  $V$ . Let  $p$  be a prime such that

$$L_p \simeq \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \perp \langle \epsilon \rangle,$$

where  $\epsilon \in \mathbb{Z}_p^\times$ . For any nonnegative integer  $m$ , let  $\mathcal{G}_{L,p}(m)$  be a genus on a quadratic space  $W$  such that each  $\mathbb{Z}$ -lattice  $T \in \mathcal{G}_{L,p}(m)$  satisfies

$$T_p \simeq \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \perp \langle \epsilon p^m \rangle \quad \text{and} \quad T_q \simeq (L^{p^m})_q \quad \text{for any } q \neq p.$$

Here  $W = V$  if  $m$  is even,  $W = V^p$  otherwise.

In Chapter 4, we define a (multi-)graph whose vertices are ternary forms in the same genus. The graph  $\mathfrak{G}_{L,p}(m)$  is defined as follows: The set of vertices is equivalence classes in the genus  $\mathcal{G}_{L,p}(m)$ . For two equivalence classes  $[T_1]$  and  $[T_2]$  are connected by an edge if there is a  $\mathbb{Z}$ -lattice  $T' \in [T_1]$  and a basis  $\{x_1, x_2, x_3\}$  of  $W$  such that

$$T' = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3 \quad \text{and} \quad T_2 = \mathbb{Z}(px_1) + \mathbb{Z}\left(\frac{x_2}{p}\right) + \mathbb{Z}x_3.$$

This finite graph is closely related with the infinite graph defined by Benham and Hsia in [1]. We prove various properties of this graph that are needed in the next chapter. In particular, we give a relation between the incidence matrix of this graph and Eichler's Anzahlmatrix.

The aim of Chapter 5 is to show that if  $T \in \mathcal{G}_{L,p}(m)$  for  $m = 0$  or  $1$ , then

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there are rational numbers  $a_i, b_i$  such that

$$r(n, T) = \sum_{[S_i] \in \mathcal{G}_{L,p}(m+1)} (a_i r(pn, S_i) + b_i r(p^3 n, S_i)) + (\text{some extra term}).$$

We prove this statement in each case and compute the rational numbers  $a_i$ 's,  $b_i$ 's and the extra term explicitly. For the case when  $m = 2$ , we give an example such that the above statement does not hold, and prove that the above statement still holds for  $m = 2$  if we additionally assume that  $n$  is divisible by  $p$ . In the case when  $m \geq 3$ , we show that under some restriction, the above statement holds if we replace  $r(n, T)$  by  $r(p^2 n, T) - pr(n, T)$ , and for any integer  $n$  not divisible by  $p$ , both  $r(n, T)$  and  $r(pn, T)$  can be written as a linear summation of  $r(pn, S)$ 's and  $r(n, S)$ 's, respectively, for  $S \in \mathcal{G}_{L,p}(m+1)$ . In some cases, the extra term in the above equation can be removed. To determine when it happens, we need to know some structure of the graph  $\mathfrak{G}_{L,p}(m)$ .

In Chapter 6, we discuss genus-correspondences between ternary quadratic forms respecting spinor genus. Let  $n$  be a positive integer. Let  $M, N$  be ternary  $\mathbb{Z}$ -lattices such that there is a representation  $\phi : M^n \rightarrow N$  such that  $[N : \phi(M^n)] = n$ . For two  $\mathbb{Z}$ -lattices  $M' \in \text{gen}(M)$  and  $N' \in \text{gen}(N)$ , assume that  $(M')^n$  is represented by  $N'$ . Under this situation, Jagy conjectured in [12] that if  $g^+(M) = g^+(N)$ , then  $M' \in \text{spn}(M)$  if and only if  $N' \in \text{spn}(N)$ . In fact, this conjecture is not true. We slightly modify this conjecture and prove it. We also construct genus-correspondences satisfying some additional properties. In particular, we construct infinite family of genera of ternary quadratic forms that possess (absolutely) complete systems of spinor exceptional integers.

# Chapter 2

## Preliminaries

In this chapter, we introduce some definitions and properties which are needed in this thesis. Any unexplained notations and terminologies can be found in [15] or [17].

### 2.1 Definitions

Let  $\mathbb{Q}$  be the field of rational numbers and  $\Omega$  be the set of all primes including  $\infty$ . For a prime  $p \in \Omega$ , we denote the fields of  $p$ -adic completions of  $\mathbb{Q}$  by  $\mathbb{Q}_p$ , in particular,  $\mathbb{Q}_\infty$  means the field of real numbers  $\mathbb{R}$ . Let  $F$  be a field  $\mathbb{Q}$  or  $\mathbb{Q}_p$ . A *quadratic space*  $V$  is a vector space over  $F$  of finite rank equipped with a non-degenerate symmetric bilinear form  $B : V \times V \rightarrow F$ . So we have

$$\begin{aligned} B(x, y + z) &= B(x, y) + B(x, z) \\ B(\alpha x, y) &= \alpha B(x, y), \quad B(x, y) = B(y, x) \end{aligned}$$

for any  $x, y, z \in V$  and  $\alpha \in F$ . We put the *quadratic map*  $Q(x) = B(x, x)$  for any  $x \in V$ . Then the following equalities hold:

$$\begin{aligned} Q(\alpha x) &= \alpha^2 Q(x) \\ Q(x + y) &= Q(x) + Q(y) + 2B(x, y) \end{aligned}$$

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for any  $x, y \in V$  and  $\alpha \in F$ . For a non-zero  $\alpha \in F$ , we denote by  $V^\alpha$  the quadratic space obtained from scaling  $V$  by  $\alpha$ .

For two quadratic spaces  $V$  and  $W$ , a linear map  $\sigma : V \rightarrow W$  is called a *representation* of  $V$  into  $W$  if

$$B(x, y) = B(\sigma(x), \sigma(y))$$

for any  $x, y \in V$ . In this case we say that  $V$  is represented by  $W$ . Furthermore if  $\sigma$  is bijective,  $\sigma$  is called an *isometry* of  $V$  onto  $W$ . In this case, we say  $V$  is isometric to  $W$  and denote  $V \simeq W$ . An *isometry group*  $O(V)$  is the set of all isometries  $V$  onto  $V$ . We call  $\sigma$  a *rotation* if  $\det \sigma = 1$ , and we define  $O^+(V)$  be the set of all rotations of  $V$ .

Let  $x_1, \dots, x_n$  be a basis of a quadratic space  $V$  over  $F$ . The determinant

$$\det(B(x_i, x_j))$$

of the  $n \times n$  matrix  $(B(x_i, x_j))$  is called the *discriminant* of the quadratic space  $V$  and denoted by  $dV$ . Note that the discriminant  $dV$  is uniquely determined up to  $(F^\times)^2$ .

Let  $V$  be a  $n$ -ary quadratic space over  $F$  with the quadratic map  $Q$ . Any  $\sigma \in O^+(V)$  can be expressed as a product of symmetries by Theorem 43:4 in [17], say

$$\sigma = \tau_{v_1} \cdots \tau_{v_r}.$$

We can attach a well-defined invariant to the isometry  $\sigma$ , namely the canonical image of  $Q(v_1) \cdots Q(v_r)$  in  $F^\times / (F^\times)^2$ . We call this invariant the *spinor norm* of  $\sigma$  and write

$$\theta(\sigma) \in F^\times / (F^\times)^2.$$

So we have a group homomorphism

$$\theta : O^+(V) \rightarrow F^\times / (F^\times)^2.$$

The kernel of the homomorphism  $\theta$  is denoted by  $O'(V)$ . Thus

$$O'(V) = \{\sigma \in O^+(V) \mid \theta(\sigma) = 1\}.$$

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Let  $L_F(V)$  be the algebra of linear transformations of  $V$  into it self and  $e_1, \dots, e_n$  be a fixed basis of  $V$ . Consider an element  $\sigma \in L_F(V)$ , write

$$\sigma e_j = \sum_i \alpha_{ij} e_i \quad (\alpha_{ij} \in F)$$

for  $1 \leq j \leq n$  and define the norm of  $\sigma$  as following:

$$\|\sigma\|_p = \max\{|\alpha_{ij}|_p \mid 1 \leq i, j \leq n\}.$$

where  $|\cdot|_p$  is the given  $p$ -adic valuation on  $F$ .

For a quadratic space  $V$  over  $\mathbb{Q}$  and a finite prime  $p$ ,  $V_p := \mathbb{Q}_p \otimes V$  is a quadratic space over  $\mathbb{Q}_p$ . We fix the basis for the quadratic space  $V$  and assume each norm  $\|\cdot\|_p$  on  $L_{F_p}(V_p)$  is defined with respect to this fixed basis.

**Theorem 2.1.1. Weak Approximation Theorem for Rotations.** *Let  $V$  be a quadratic space over  $\mathbb{Q}$  and  $T$  be a finite set of primes. Suppose  $\phi_p$  is given in  $O^+(V_p)$  at each prime  $p$  in  $T$ . Then for each  $\epsilon > 0$  there is a  $\sigma \in O^+(V)$  such that*

$$\|\sigma - \phi_p\|_p < \epsilon$$

for any  $p \in T$ .

*Proof.* See 101:7 in [17]. □

**Theorem 2.1.2. Strong Approximation Theorem for Rotations.** *Let  $V$  be a quadratic space over  $\mathbb{Q}$  with  $\dim(V) \geq 3$ ,  $S$  be an indefinite set of primes for  $V$ , and  $T$  be a finite subset of  $S$ . A rotation  $\phi_p$  is given in  $O'(V_p)$  at each prime  $p$  in  $T$ . Then for each  $\epsilon > 0$  there is a rotation  $\sigma$  in  $O'(V)$  such that*

$$\|\sigma - \phi_p\|_p < \epsilon$$

for any  $p \in T$  and

$$\|\sigma\|_p = 1$$

for any  $p \in S - T$ .

*Proof.* See 104:4 in [17]. □



## CHAPTER 2. PRELIMINARIES

Consider the product

$$\prod_{p \in \Omega} O^+(V_p).$$

An element of this group is defined coordinatewise, say

$$\Sigma = (\Sigma_p)_{p \in \Omega} \quad (\Sigma_p \in O^+(V_p)).$$

For two such elements  $\Sigma, \Lambda$  we have following equalities:

$$(\Sigma\Lambda)_p = \Sigma_p\Lambda_p, \quad (\Sigma^{-1})_p = \Sigma_p^{-1},$$

for any  $p \in \Omega$ . An element  $\Sigma$  of the above product is called a *split rotation* of the quadratic space  $V$  if  $\Sigma$  satisfies the property

$$\|\Sigma_p\|_p = 1 \text{ for almost all } p.$$

The definition is independent of choices of the basis for  $V$ . We denote the set of all split rotations by  $J_V$ . It is a subgroup of the above product and called the group of split rotations. The set of all split rotations  $\Sigma$  satisfying the property

$$\Sigma_p \in O'(V_p)$$

for any prime  $p$  is denoted by  $J'_V$ . Clearly  $J'_V$  is a subgroup of  $J_V$ .

For a finite prime  $p$ , the ring of  $p$ -adic integers is denoted by  $\mathbb{Z}_p$ . Let  $R$  be the ring of integers  $\mathbb{Z}$  or ring of  $p$ -adic integers  $\mathbb{Z}_p$  and  $F$  be the quotient field of  $R$ . Let  $L$  be a subset of the quadratic space  $V$  over  $F$  that is an  $R$ -module under the laws induced from the vector space  $V$  over  $F$ . We define a subspace  $FL$  of  $V$  as follows:

$$FL = \{\alpha x \mid \alpha \in F, x \in L\}.$$

We call the  $R$ -module  $L$  a *lattice* on  $V$  if there is a basis  $x_1, \dots, x_n$  for  $V$  such that

$$L \subseteq Rx_1 + \dots + Rx_n$$

and  $FL = V$ . In particular, we say  $L$  is a  $\mathbb{Z}$ -lattice ( $\mathbb{Z}_p$ -lattice) on  $V$  if  $R = \mathbb{Z}$  ( $\mathbb{Z}_p$ , respectively). Sometimes, we omit the term “on  $V$ ” for convenience.

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Let  $e_1, e_2, \dots, e_n$  be a basis of the lattice  $L$  on  $V$ . We define the symmetric matrix  $M_L$  by  $(B(e_i, e_j))$ , where  $B$  is the symmetric bilinear form defined on  $V$ . We define the *discriminant*  $dL$  of  $L$  by the determinant of  $M_L$ . We say  $L$  is *positive definite* (*indefinite*) if  $M_L$  is a positive definite (indefinite, respectively) matrix. We define the *scale* of  $L$  to be the  $R$ -module generated by  $B(x, y)$  for all  $x, y \in L$ , *norm* of  $L$  to be the  $R$ -module generated by  $Q(x)$  for all  $x \in L$ , which are denoted by  $\mathfrak{s}(L)$  and  $\mathfrak{n}(L)$ , respectively. Note that  $\mathfrak{s}(L)$  and  $\mathfrak{n}(L)$  are either a fractional ideal or 0. For  $\alpha \in R^\times$ ,  $L^\alpha$  means the lattice  $L$  which is regarded as a lattice on  $V^\alpha$ .

Let  $U, V$  be quadratic spaces over  $F$ . Consider lattices  $K$  on  $U$  and  $L$  on  $V$ . We say that  $K$  is *represented* by  $L$ , if there is a representation  $\sigma : FK \rightarrow FL$  such that  $\sigma K \subseteq L$ . We say that  $K$  and  $L$  are *isometric*, if there is an isometry  $\sigma : FK \rightarrow FL$  such that  $\sigma K = L$ , in this case we write

$$K \simeq L.$$

Let  $K, L$  be lattices on  $V$ . We say that  $K$  and  $L$  are in the same *class* if  $K$  and  $L$  are isometric. This is clearly an equivalence relation on the set of all lattices on  $V$ . We use

$$\text{cls}(L)$$

to denote the class of  $L$ . We define the subgroup  $O(L)$  of  $O(V)$  as follows:

$$O(L) = \{\sigma \in O(V) \mid \sigma L = L\}.$$

We denote the order  $|O(L)|$  of  $O(L)$  by  $o(L)$ . And define

$$O^+(L) = O(L) \cap O^+(V).$$

For a  $\mathbb{Z}$ -lattice  $L$  on  $V$ ,  $L_p$  means a  $\mathbb{Z}_p$ -lattice  $\mathbb{Z}_p \otimes L$  on  $V_p$ . The *genus*  $\text{gen}(L)$  of the  $\mathbb{Z}$ -lattice  $L$  on  $V$  is the set of all  $\mathbb{Z}$ -lattices  $K$  on  $V$  satisfies the following property: for each finite prime  $p$  there is a isometry  $\Sigma_p \in O(V_p)$  such that

$$K_p = \Sigma_p L_p.$$

The genus can be described in terms of split rotations:  $K$  belongs to  $\text{gen}(L)$

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if and only if

$$K = \Sigma L$$

for some  $\Sigma \in J_V$ . We say that the  $\mathbb{Z}$ -lattice  $K$  on  $V$  is contained in the same *spinor genus* as  $L$  if there is an isometry  $\sigma \in O(V)$  and a rotation  $\Sigma_p \in O'(V_p)$  at each finite prime  $p$  such that

$$K_p = \sigma_p \Sigma_p L_p$$

for any finite prime  $p$ . This condition can be expressed in terms of split rotations: there is a  $\sigma \in O(V)$  and a  $\Sigma \in J'_V$  such that

$$K = \sigma \Sigma L.$$

We denote the set of all lattices in the same spinor genus as  $L$  by  $\text{spn}(L)$ . Then we have

$$\text{cls}(L) \subseteq \text{spn}(L) \subseteq \text{gen}(L).$$

The number of classes and spinor genera in  $\text{gen}(L)$  is denoted by  $h(L)$  and  $g(L)$ , respectively. Note that  $h(L)$  and  $g(L)$  are always finite.

## 2.2 Spinor norms of local integral rotations

Consider the product

$$\prod_{p \in \Omega} \mathbb{Q}_p^\times.$$

An element of this group is defined in terms of its  $p$ -coordinates, say

$$\mathbf{i} = (\mathbf{i}_p)_{p \in \Omega} \quad (\mathbf{i}_p \in \mathbb{Q}_p^\times);$$

and the multiplication in the product is defined coordinatewise. An element of the above product is called *idèle* if it satisfies the following condition:

$$|\mathbf{i}_p|_p = 1 \text{ for almost all } p \in \Omega.$$

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The set of all idèles is a subgroup of the product called the group of idèles and denoted by  $J_{\mathbb{Q}}$ . Let  $D$  be the set of all positive rational numbers and  $P_D$  be the group of *principal idèles* of the form  $(\alpha)_{p \in \Omega}$  with  $\alpha \in D$ . For a  $\mathbb{Z}$ -lattice  $L$ , we define the subgroup  $J_{\mathbb{Q}}^L$  of  $J_{\mathbb{Q}}$  as follows:

$$J_{\mathbb{Q}}^L = \{\mathfrak{i} \in J_{\mathbb{Q}} \mid \mathfrak{i}_p \in \theta(O^+(L_p)) \text{ for any finite prime } p\}.$$

Let  $K$  and  $L$  be  $\mathbb{Z}$ -lattices on  $V$ . We say that  $K$  is contained in the same *proper spinor genus* as  $L$  if there is a rotation  $\sigma \in O^+(V)$  and a rotation  $\Sigma_p \in O'(V_p)$  at each finite prime  $p$  such that

$$K_p = \sigma_p \Sigma_p L_p$$

for any finite prime  $p$ . This condition can be expressed in terms of split rotations : there is a  $\sigma \in O^+(V)$  and a  $\Sigma \in J'(V)$  such that  $K = \sigma \Sigma L$ . We denote the set of all lattices in the same proper spinor genus as  $L$  by  $\text{spn}^+(L)$ . The number of proper spinor genera in  $\text{gen}(L)$  is denoted by  $g^+(L)$ . The number  $g^+(L)$  can be computed by means of an idèlic index formula:

**Theorem 2.2.1.** *For a  $\mathbb{Z}$ -lattice  $L$ , assume that  $\text{rank}(L) \geq 3$ . Then*

$$g^+(L) = (J_{\mathbb{Q}} : P_D J_{\mathbb{Q}}^L).$$

*Proof.* See 102:7 in [17]. □

Above theorem shows that to compute the number of proper spinor genera in the given genus, it is necessary to compute the spinor norm of local integral rotations at each finite prime  $p$ . For this, we introduce some notations and definitions. Let  $\mathfrak{U}_p$  be the group of units in  $\mathbb{Z}_p$  and  $\Delta = 1 + 4\rho$  be a non-square unit of quadratic defect  $4\mathbb{Z}_p$ . We use the symbol  $A(\alpha, \beta)$  to denote the  $2 \times 2$  matrix

$$A(\alpha, \beta) = \begin{pmatrix} \alpha & 1 \\ 1 & \beta \end{pmatrix},$$

where  $\alpha, \beta \in \mathbb{Z}$ . For any lattice  $L$ , we define

$$P(L) = \{v \in L \mid v \text{ maximal and } \tau_v \in O(L)\}.$$

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Let  $L$  be a  $\mathbb{Z}_p$ -lattice of rank  $n$  and  $p$  be a prime. At first, we consider modular cases. We may assume that  $L$  is unimodular by scaling. If  $n = 1$ , then clearly  $\theta(O^+(L)) = (\mathbb{Q}_p^\times)^2$  for any prime  $p$ . When  $p$  is an odd prime,  $\theta(O^+(L)) = \mathfrak{U}_p(\mathbb{Q}_p^\times)^2$  for  $n \geq 2$  (See 92:5 in [17]). So we have only to consider the 2-adic cases. The following two theorems complete the answer for 2-adic cases.

**Theorem 2.2.2.** *Let  $L$  be a unimodular  $\mathbb{Z}_2$ -lattice with  $\text{rank}(L) = 2$ . If  $L$  is isometric to one of the lattices  $A(0,0)$ ,  $A(2,2\rho)$ ,  $A(1,0)$  and  $A(1,4\rho)$ , then  $\theta(O^+(L)) = \mathfrak{U}_2(\mathbb{Q}_2^\times)^2$ . If  $L$  is isometric to  $A(c,2f)$ , where  $c, f$  are units, then  $\theta(O^+(L)) = Q(\langle 1, d \rangle)(\mathbb{Q}_2^\times)^2$  where  $d = d(A(c,2f))$ .*

*Proof.* See Remark (a) in [10]. □

**Theorem 2.2.3.** *Let  $L$  be a unimodular  $\mathbb{Z}_2$ -lattice with  $\text{rank}(L) \geq 3$ . Then,  $\theta(O^+(L)) = \mathfrak{U}_2(\mathbb{Q}_2^\times)^2$  if and only if  $\mathfrak{n}(L) = 2\mathbb{Z}_2$ . If  $\mathfrak{n}(L) = \mathbb{Z}_2$ , then  $\theta(O^+(L)) = \mathbb{Q}_2^\times$ .*

*Proof.* See Proposition A in [10]. □

Next, we determine the spinor norm  $\theta(O^+(L))$  for arbitrary  $\mathbb{Z}_p$ -lattice  $L$ . If  $p$  is an odd prime, we can directly compute the spinor norm  $\theta(O^+(L))$  from the following lemma.

**Lemma 2.2.4.** *Assume  $p$  is an odd prime and  $L$  is a  $\mathbb{Z}_p$ -lattice. For a primitive vector  $x$  in  $L$ , the symmetry  $\tau_x$  is contained in  $O(L)$  if and only if  $Q(x)$  splits  $L$ .*

*Proof.* The symmetry  $\tau_x$  is contained in  $O(L)$  if and only if

$$\frac{2B(x, L)}{Q(x)} \subseteq \mathbb{Z}_p.$$

From the above, the lemma is proved immediately. □

In the remaining of this section, we determine the spinor norm of the group of local integral rotations for the arbitrary 2-adic lattices. Assume  $L$  is a binary  $\mathbb{Z}_2$ -lattice. If  $L$  is a modular lattice, then  $\theta(O^+(L))$  is already

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determined. So we deal only with the non-modular case. By scaling we may assume that

$$L \simeq \langle 1 \rangle \perp \langle 2^r \alpha \rangle,$$

where  $r \geq 1$  and  $\alpha \in \mathfrak{U}_2$ .

**Theorem 2.2.5.** *Under the above assumptions,  $\theta(O^+(L))$  is completely determined as follows:*

$$\theta(O^+(L)) = \begin{cases} \{\gamma \in \mathbb{Q}^\times \mid (\gamma, -2\alpha) = +1\} & \text{if } r = 1, 3, \\ \{\gamma \in \mathfrak{U}_2(\mathbb{Q}_2^\times)^2 \mid (\gamma, -\alpha) = +1\} & \text{if } r = 2, \\ (\mathbb{Q}_2^\times)^2 \cup \alpha(\mathbb{Q}_2^\times)^2 \cup \Delta(\mathbb{Q}_2^\times)^2 \cup \alpha\Delta(\mathbb{Q}_2^\times)^2 & \text{if } r = 4, \\ (\mathbb{Q}_2^\times)^2 \cup 2^r \alpha(\mathbb{Q}_2^\times)^2 & \text{if } r \geq 5. \end{cases}$$

*Proof.* See [6]. □

Next, we consider the higher dimensional cases with 1-dimensional components. Let  $L$  be a  $\mathbb{Z}_2$ -lattice of rank  $\geq 3$  with Jordan decomposition

$$L = \langle 1 \rangle \perp \langle 2^{r_2} \alpha_2 \rangle \perp \cdots \perp \langle 2^{r_n} \alpha_n \rangle,$$

where  $r_i \in \mathbb{Z}$ ,  $\alpha_i \in \mathfrak{U}_2$  for  $i = 1, \dots, n$  and  $r_i < r_{i+1}$  for  $i = 1, \dots, n-1$ ,  $r_1 = 0$ . For  $j = 1, \dots, n-1$ , we define

$$\begin{aligned} L_{j,j+1} &= \langle 2^{r_j} \alpha_j \rangle \perp \langle 2^{r_{j+1}} \alpha_{j+1} \rangle, \\ r(L_{j,j+1}) &= r_{j+1} - r_j \end{aligned}$$

**Theorem 2.2.6.** *Suppose that there is at least one  $k$  for which  $r(L_{k,k+1}) = 1$  or 3. If  $r_s - r_t = 2$  or 4 for some  $s, t = 1, \dots, n$ , we have  $\theta(O^+(L)) = \mathbb{Q}_2^\times$ .*

*Proof.* See Theorem 2.2 in [6]. □

**Theorem 2.2.7.** *Suppose that  $L$  does not satisfy the hypothesis of Theorem 2.2.6. Then*

$$\theta(O^+(L)) = \left\{ \prod_{i=1}^{\text{even}} Q(v_i) \mid v_i \in P(L_{j_i, j_{i+1}}), 1 \leq j_i \leq n-1 \right\}.$$

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*Proof.* See Theorem 2.7 in [6].  $\square$

Finally, we consider the higher dimensional cases with arbitrary components. Assume that  $L$  is a  $\mathbb{Z}_2$ -lattice of rank  $\geq 3$  with at least one Jordan component of rank  $\geq 2$ .

**Definition 2.2.8.** We say  $\mathbb{Z}_2$ -lattice  $L$  has *even order* if  $Q(P(L)) \subseteq \mathfrak{U}_2(\mathbb{Q}_2^\times)^2$  and  $L$  has *odd order* if  $Q(P(L)) \subseteq 2\mathfrak{U}_2(\mathbb{Q}_2^\times)^2$ .

**Proposition 2.2.9.** *Let  $L$  be unimodular.*

(1) *Suppose  $\text{rank}(L) = 2$ . Then*

$$\begin{aligned} L \text{ has odd order} &\iff L \simeq A(0, 0) \text{ or } A(2, 2\rho), \\ L \text{ has even order} &\iff L \simeq A(1, 0) \text{ or } A(1, 4\rho). \end{aligned}$$

(2) *Suppose  $\text{rank}(L) \geq 3$ . Then*

$$\theta(O^+(L)) \neq \mathbb{Q}_2^\times \implies L \text{ has odd order.}$$

*Proof.* See Proposition 3.2 in [6].  $\square$

**Theorem 2.2.10.** *Let  $L = 2^{r_1}L_1 \perp 2^{r_2}L_2 \perp \cdots \perp 2^{r_t}L_t$ , where  $L_i$  is unimodular and  $0 = r_1 < r_2 < \cdots < r_t$  are natural numbers. Then we have:*

- (a) *If at least one Jordan component has rank  $\geq 3$ , then  $\theta(O^+(L)) \neq \mathbb{Q}^\times$  (in fact,  $= \mathfrak{U}_2(\mathbb{Q}_2^\times)^2$ ) if and only if all  $2^{r_i}L_i$  have the same order for  $i = 1, \dots, t$ .*
- (b) *If  $\text{rank}(L_i) \leq 2$  for every  $i$  of which at least one component, say  $L_{i_0}$ , is binary, then  $\theta(O^+(L)) \neq \mathbb{Q}_2^\times$  if and only if one of the following three cases occurs:*
  - (1) *all Jordan components have odd order,*
  - (2) *all Jordan components have even order,*
  - (3) *whenever  $\text{rank}(L_i) = 2$ ,  $L_i \simeq A(a_i, 2b_i)$ ,  $a_i, b_i \in \mathfrak{U}_2$ ; moreover,*
    - (i) *the associated spaces of all binary components are isometric,*

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- (ii) for any unitary component, say  $L_{i'} \simeq \langle c_{i'} \rangle$ ,  $c_{i'} \in \mathfrak{U}_2$ , the Hilbert symbol  $(2^{r_{i'}-r_{i_0}} a_{i_0} c_{i'}, -dL_{i_0}) = 1$ ,
- (iii)  $r_{j+1} - r_j \geq 4$ , or  $r_{j+1} - r_j = 2$  with  $\text{rank}(L_j) = \text{rank}(L_{j+1}) = 1$  and  $d(2^{r_j} L_j \perp 2^{r_{j+1}} L_{j+1}) \cdot d(2^{r_{i_0}} L_{i_0}) \in \{(\mathbb{Q}_2^\times)^2, \Delta(\mathbb{Q}_2^\times)^2\}$  for  $j = 1, \dots, t-1$ .

Finally, in case (1) and (2),  $\theta(O^+(L)) = \mathfrak{U}_2(\mathbb{Q}_2^\times)^2$ ; and in case (3) it is equal to  $\theta(O^+(L_{i_0})) = \{c \in \mathbb{Q}_2^\times \mid (c, -dL_{i_0}) = 1\}$ .

*Proof.* See Theorem 3.14 in [6], 1.2 in [7] and [4].  $\square$

### 2.3 Spinor exceptional integers

Let  $L$  be an integral  $\mathbb{Z}$ -lattice and  $a$  be an integer. Assume  $\text{rank}(L) \geq 4$ . If  $a$  is represented by  $\text{gen}(L)$  (i.e., there is a lattice in the genus of  $L$  representing  $a$ ), then  $a$  is represented by every spinor genus in genus of  $L$  (i.e., there is a lattice in each spinor genus representing  $a$ ). However this does not hold when  $\text{rank}(L) = 3$ . Assume that  $\text{rank}(L) = 3$ . If  $a$  is represented by  $\text{gen}(L)$ , then either every spinor genus in the genus of  $L$  represents  $a$  or precisely half of all the spinor genera do. In the latter case,  $c$  is called a *spinor exceptional integer* for  $\text{gen}(L)$  and the half-genus that represents (doesn't represent)  $c$  is called *good* (*bad*, respectively) half-genus with respect to  $c$  (see [1]).

Let  $L$  be a ternary  $\mathbb{Z}$ -lattice on  $V$ . Recall that  $J_{\mathbb{Q}}$  is the group of idèles,  $J_V$  is the group of split rotations and  $P_D$  is the group of principal idèles, where  $D$  is the set of all positive rational numbers. We define the subgroup  $J_L$  of  $J_V$  by the equation

$$J_L = \{\Sigma \in J_V \mid \sigma_p \in O^+(L_p) \text{ for any finite prime } p\}.$$

For a split rotation  $\Sigma = (\Sigma_p)_{p \in \Omega}$  in  $J_V$ ,  $\theta(\Sigma)$  is the set of idèles  $\mathbf{i} = (\mathbf{i}_p)_{p \in \Omega}$  with  $\mathbf{i}_p \in \theta(\Sigma_p)$  for any prime  $p \in \Omega$  ( $\theta$  is spinor norm) and  $\theta(J_L)$  is the union of all  $\theta(\Sigma)$  with  $\Sigma \in J_L$ .

**Theorem 2.3.1.** *Let  $L$  be a ternary  $\mathbb{Z}$ -lattice on  $V$ . Assume that a non-zero integer  $a$  is represented by  $\text{gen}(L)$ . Let  $d = -a \cdot dV$ ,  $E = \mathbb{Q}(\sqrt{d})$ . Then a*



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is either represented by all spinor genera in the genus of  $L$  of exactly half of them. The latter case occurs only when the followings hold:

$$a \neq 0, \quad d \notin (\mathbb{Q}^\times)^2, \quad \theta(J_L) \subseteq N_{E/\mathbb{Q}}(J_E). \quad (2.3.1)$$

If these conditions are satisfied, the genus of  $L$  decomposed into two half genera such that for  $\Sigma \in J_V$ ,  $L$  and  $\Sigma L$  lie in the same half genus if and only if  $\theta(\Sigma) \subseteq P_D N_{E/\mathbb{Q}}(J_E)$ . A half genus consists of spinor genera; two spinor genera in the same half genus represents  $a$  or not simultaneously.

*Proof.* See Satz 1 in [19]. □

We now investigate when an integer  $a$  which satisfies the condition (2.3.1) is actually a spinor exceptional integer of  $L$ .

**Definition 2.3.2.** Let  $p$  be a finite prime,  $a \in Q(L_p)$  and  $x \in L_p$  with  $Q(x) = a$ . Then the subgroup of  $\mathbb{Q}_p^\times$  generated by

$$\{c \in \mathbb{Q}_p^\times \mid \text{there is } \phi \in O^+(V_p) \text{ with } x \in \phi(L_p) \text{ and } c \in \theta(\phi)\}$$

is denoted by  $\theta(L_p, a)$ .

Note that  $\theta(L_p, a)$  is independent of choices of  $x$  (see [19]).

**Theorem 2.3.3.** Let  $L, a, d, E$  be as in Theorem 2.3.1. For any prime  $p \in \Omega$ , we denote  $N_{E_{\mathfrak{P}}/\mathbb{Q}_p}(E_{\mathfrak{P}}^\times)$  by  $N_p(E)$ , where  $\mathfrak{P}$  is an extension of  $p$  to  $E$ . Then  $a$  is a spinor exceptional integer for genus of  $L$  exactly when the condition (2.3.1) is satisfied and also

$$\theta(L_p, a) = N_p(E) \quad (2.3.2)$$

holds for any finite prime  $p$ .

*Proof.* See Satz 2 in [19]. □

To apply Theorem 2.3.3, it is need to compute the  $\theta(L_p, a)$ . Let  $L$  be a ternary  $\mathbb{Z}$ -lattice and  $p$  be a finite prime. Assume that  $\mathfrak{s}L_p = \mathbb{Z}_p$ . Furthermore we assume,  $a \in Q(L_p)$ ,  $a \neq 0$ ,  $d, E$  as in Theorem 2.3.1,  $\mathfrak{P}$  is an

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extension of  $p$  to  $E$ ,  $N_p(E)$  as in Theorem 2.3.3 and  $\theta(O^+(L_p)) \subseteq N_p(E)$ . In addition, we assume  $d \notin (\mathbb{Q}_p^\times)^2$ , otherwise the condition 2.3.2 is trivially satisfied. We denote the quadratic defect of an element  $\xi$  of  $\mathbb{Q}_2$  by  $\mathfrak{d}(\xi)$ .

**Theorem 2.3.4.** *Let  $p$  be an odd prime.*

(a) *If  $E_{\mathfrak{P}}/\mathbb{Q}_p$  is unramified,  $\theta(O^+(L_p)) \subseteq N_p(E)$  if and only if*

$$L_p \simeq \langle b_1 \rangle \perp \langle p^{2r}b_2 \rangle \perp \langle p^{2s}b_3 \rangle \quad (b_i \in \mathfrak{U}_p, \ 0 \leq r \leq s).$$

*Then  $\theta(L_p, a) \neq N_p(E)$  if and only if one of the following conditions is satisfied:*

- (i)  $-b_1b_2 \in (\mathbb{Q}_p^\times)^2$  and  $\text{ord}_p(a) \geq 2r + 1$ ,
- (ii)  $-b_1b_2 \notin (\mathbb{Q}_p^\times)^2$  and  $\text{ord}_p(a) \geq 2s + 1$ .

(b) *If  $E_{\mathfrak{P}}/\mathbb{Q}_p$  is ramified, it follows from  $\theta(O^+(L_p)) \subseteq N_p(E)$  that*

$$L_p \simeq \langle b_1 \rangle \perp \langle p^r b_2 \rangle \perp \langle p^s b_3 \rangle \quad (b_i \in \mathfrak{U}_p, \ 0 < r < s).$$

*Then  $\theta(L_p, a) \neq N_p(E)$  if and only if one of the following conditions holds:*

- (i)  $r$  is even and  $\text{ord}_p(a) \geq r$ ,
- (ii)  $r$  is odd and  $\text{ord}_p(a) \geq s$ .

*Proof.* See Satz 3 in [19]. □

**Theorem 2.3.5.** *Let  $p$  be an even prime.*

(a) *If  $E_{\mathfrak{P}}/\mathbb{Q}_p$  is unramified, then  $\theta(O^+(L_2)) \subseteq N_2(E)$  if and only if  $L_2$  is not unimodular and either its all Jordan components have even order or have odd order. In this case,  $\theta(L_2, a) \neq N_2(E)$  if and only if one of the following conditions is satisfied:*

- (i)  $L_2 \simeq \langle b_1 \rangle \perp \langle 2^{2r}b_2 \rangle \perp \langle 2^{2s}b_3 \rangle$  ( $b_i \in \mathfrak{U}_2$ ,  $0 \leq r \leq s$ ) and
  - ( $\alpha$ )  $\mathfrak{d}(-b_1b_2) = 2\mathbb{Z}_2$ ,  $\text{ord}_2(a) \geq 2r$ , or
  - ( $\beta$ )  $\mathfrak{d}(-b_1b_2) = 4\mathbb{Z}_2$ ,  $\text{ord}_2(a) \geq 2s$ , or

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$$(\gamma) \quad \mathfrak{d}(-b_1 b_2) = 0, \text{ ord}_2(a) \geq 1 + 2r + 2t \quad (t := \min(1, s - 1)).$$

(ii)  $L_2$  does not have an 1-dimensional orthogonal decomposition and

$$(\alpha) \quad L_2 \simeq A(0, 0) \perp \langle 2^{2r+1}b \rangle \quad (b \in \mathfrak{U}_2, \quad r \geq 0), \quad \text{ord}_2(a) \geq 1 + s \\ (s := \min(1, r)), \text{ or}$$

$$(\beta) \quad L_2 \simeq A(2, 2\rho) \perp \langle 2^{2r+1}b \rangle \quad (b \in \mathfrak{U}_2, \quad r \geq 0), \quad \text{ord}_2(a) \geq 2r + 1, \text{ or}$$

$$(\gamma) \quad L_2 \simeq \langle b \rangle \perp M \quad (b \in \mathfrak{U}_2, \quad M \text{ is } 2^{2r+1}\text{-modular with even order,} \\ r \geq 0), \quad \text{ord}_2(a) \geq 2r + 2.$$

(b) If the extension  $E_{\mathfrak{P}}/\mathbb{Q}_p$  is ramified and  $\text{ord}_2(d)$  is even, it follows from  $\theta(O^+(L_2)) \subseteq N_2(E)$  that

$$L_2 \simeq \langle b_1 \rangle \perp \langle 2^r b_2 \rangle \perp \langle 2^s b_3 \rangle \quad (b_i \in \mathfrak{U}_2, \quad 0 \leq r \leq s).$$

In this case,  $\theta(L_2, a) \neq N_2(E)$  if and only if one of the followings is satisfied with

$$K := \langle 2^{r-2}b_1 \rangle \perp \langle 2^r b_2 \rangle \perp \langle 2^s b_3 \rangle,$$

$$K' := \langle 2^r b_1 \rangle \perp \langle 2^r b_2 \rangle \perp \langle 2^s b_3 \rangle :$$

$$(i) \quad r \text{ is even, } \theta(O^+(K)) \not\subseteq N_2(E), \quad \text{ord}_2(a) \geq r - 2,$$

$$(ii) \quad r \text{ is even, } \theta(O^+(K)) \subseteq N_2(E), \quad \theta(O^+(K')) \not\subseteq N_2(E), \quad \text{ord}_2(a) \geq r,$$

$$(iii) \quad r \text{ is even, } \theta(O^+(K)) \subseteq N_2(E), \quad \theta(O^+(K')) \subseteq N_2(E), \quad \text{ord}_2(a) \geq s - 2,$$

$$(iv) \quad r \text{ is odd, } \text{ord}_2(a) \geq r - 3.$$

(c) If the extension  $E_{\mathfrak{P}}/\mathbb{Q}_p$  is ramified and  $\text{ord}_2(d)$  is odd, it follow from  $\theta(O^+(L_2)) \subseteq N_2(E)$  that

$$L_2 \simeq \langle b_1 \rangle \perp \langle 2^r b_2 \rangle \perp \langle 2^s b_3 \rangle \quad (b_i \in \mathfrak{U}_2, \quad 0 < r < s)$$

In this case,  $\theta(L_2, a) \neq N_2(E)$  if and only if one of the following conditions holds with

$$K := \langle 2^{r-3}b_1 \rangle \perp \langle 2^r b_2 \rangle \perp \langle 2^s b_3 \rangle :$$

$$(i) \quad r \text{ is even, } \text{ord}_2(a) \geq r - 4,$$

$$(ii) \quad r \text{ is odd, } \theta(O^+(K)) \not\subseteq N_2(E), \quad \text{ord}_2(a) \geq r - 3,$$

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(iii)  $r$  is odd,  $\theta(O^+(K)) \subseteq N_2(E)$ ,  $\text{ord}_2(a) \geq s - 4$ .

*Proof.* See Satz 4 in [19].

□

# Chapter 3

## Watson transformations

In this chapter, we discuss the Watson transformations and its generalization. Let  $L$  be a non-classic integral  $\mathbb{Z}$ -lattice on the quadratic space  $V$ . For a prime  $p$ , we define

$$\Lambda_p(L) = \{x \in L \mid Q(x+z) \equiv Q(z) \pmod{p} \text{ for any } z \in L\}.$$

Let  $\lambda_p(L)$  be the primitive lattice obtained from  $\Lambda_p(L)$  by scaling  $V = L \otimes \mathbb{Q}$  by a suitable rational number. Here a  $\mathbb{Z}$ -lattice  $L$  is called *primitive*, provided  $\mathfrak{n}(L) = \mathbb{Z}$ . For general properties of  $\Lambda_p$ -transformation, see [2] and [3].

### 3.1 $H$ -type lattices

For  $L' \in \text{gen}(L)$  ( $L' \in \text{spn}(L)$ ) and any prime  $p$ , it is easy to show that  $\lambda_p(L') \in \text{gen}(\lambda_p(L))$  ( $\lambda_p(L') \in \text{spn}(\lambda_p(L))$ , respectively). It is well known that as a map,

$$\lambda_p : \text{gen}(L) \longrightarrow \text{gen}(\lambda_p(L)) \tag{3.1.1}$$

is surjective. Furthermore,  $\lambda_p(\text{spn}(L)) = \text{spn}(\lambda_p(L))$ . If we define  $\text{gen}(L)_S$  the set of all spinor genera in the  $\text{gen}(L)$ , then the map

$$\lambda_p : \text{gen}(L)_S \longrightarrow \text{gen}(\lambda_p(L))_S$$

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given by  $\text{spn}(L') \mapsto \text{spn}(\lambda_p(L'))$  for any  $\text{spn}(L') \in \text{gen}(L)_S$  is well-defined and surjective. In particular,  $g(L) \geq g(\lambda_p(L))$  for any prime  $p$ .

**Definition 3.1.1.** For a  $\mathbb{Z}$ -lattice  $L$  and a prime  $p$ , if  $g(L) = g(\lambda_p(L))$ , then we say the lattice  $L$  is of  $H$ -type at  $p$ .

From the above definition, if  $L$  is of  $H$ -type at  $p$ , then so is  $L'$  for any  $L' \in \text{gen}(L)$ .

Henceforth,  $L$  is always a positive definite non-classic integral ternary  $\mathbb{Z}$ -lattice.

**Lemma 3.1.2.** *Let  $L$  be a primitive ternary  $\mathbb{Z}$ -lattice and let  $p$  be an odd prime. Assume that after suitable scaling,*

$$L_p \simeq \langle 1, p^\alpha \epsilon_1, p^\beta \epsilon_2 \rangle,$$

where  $\alpha, \beta (\alpha \leq \beta)$  are non-negative integers and  $\epsilon_1, \epsilon_2 \in \{1, \Delta\}$ . If  $L$  is not of  $H$ -type at  $p$ , then the pairs  $(\alpha, \beta), (\epsilon_1, \epsilon_2)$  satisfy one of the conditions in Table 1.

*Proof.* By 102:7 of [17], note that

$$g^+(L) = [J_F : P_D J_F^L] \quad \text{and} \quad g^+(\lambda_p(L)) = [J_F : P_D J_F^{\lambda_p(L)}].$$

Clearly,  $\theta(O^+(\lambda_p(L)_q)) = \theta(O^+(L_q))$  for any prime  $q \neq p$ . Now one may easily check that if the pairs  $(\alpha, \beta), (\epsilon_1, \epsilon_2)$  do not satisfy one of the conditions in Table 1, then  $\theta(O^+(\lambda_p(L)_p)) = \theta(O^+(L_p))$ , which implies that the equality  $g^+(L) = g^+(\lambda_p(L))$  holds.  $\square$

Table 1 (odd case)	
$(\alpha, \beta)$	$(\epsilon_1, \epsilon_2)$
$(1, 2)$	$(1, 1)$
$(1, 2)$	$(\Delta, 1)$
$(2, k), k \geq 3$	$(1, 1)$
$(2, 2k + 1), k \geq 1$	$(1, \Delta)$

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**Lemma 3.1.3.** *Let  $L$  be a primitive ternary  $\mathbb{Z}$ -lattice. If  $L$  is not of  $H$ -type at 2, then there is an  $\eta \in \mathbb{Z}_2^\times$  such that*

$$(L^\eta)_2 \simeq \langle 1, 2^\alpha \epsilon_1, 2^\beta \epsilon_2 \rangle,$$

*and the pairs  $(\alpha, \beta)$ ,  $(\epsilon_1, \epsilon_2)$  satisfy one of the conditions in Table 2, where  $\alpha, \beta (\alpha \leq \beta)$  are non-negative integers and  $\epsilon_1, \epsilon_2 \in \mathbb{Z}_2^\times$ .*

*Proof.* The proof is quite similar to the above lemma. For the computation of the spinor norm map, see Section 2.2.  $\square$

Table 2 (Even case)

$(\alpha, \beta)$	$(\epsilon_1, \epsilon_2)$	$(\alpha, \beta)$	$(\epsilon_1, \epsilon_2)$
(0, 4)	$\epsilon_1 \equiv \epsilon_2 \equiv 1 \ (4)$	(5, 6)	$2\epsilon_1 + \epsilon_2 \in Q(\langle 1, 2\epsilon_1 \rangle)$
(1, 6)	$\epsilon_2 \in Q(\langle 1, 2\epsilon_1 \rangle)$	(5, 7)	$\epsilon_1 \epsilon_2 \equiv 1 \ (4)$
(2, 2)	$\epsilon_1 = 1, \epsilon_2 \equiv 3 \ (4)$	(5, 8)	$\epsilon_2 \equiv 2\epsilon_1 + 5 \ (8)$
(2, 4)	$\epsilon_1 \equiv 1 \ (4)$	(5, 9)	$\epsilon_1 \epsilon_2 \equiv 1 \ (4)$
(2, 6)	$\epsilon_1 \equiv 1 \ (4)$	(5, $2k$ ), $k \geq 5$	$1 + 2\epsilon_1 \not\equiv \epsilon_2 \ (8)$
(2, $2k - 1$ ), $k \geq 4$	$\epsilon_1 \equiv 2\epsilon_2 + 3 \ (8)$	(5, $2k + 1$ ), $k \geq 5$	$1 + 2\epsilon_1 \not\equiv \epsilon_1 \epsilon_2 \ (8)$
(2, $2k$ ), $k \geq 4$	$\epsilon_1 \equiv 1 \ (4)$	(6, 7)	$5 \notin Q(\langle \epsilon_1, 2\epsilon_2 \rangle)$
(3, 6)	$\epsilon_2 \equiv 1 \ (8)$	(6, 9)	$5 \notin Q(\langle \epsilon_1, 2\epsilon_2 \rangle)$
(4, 4)	$\epsilon_1 \equiv \epsilon_2 \equiv 1 \ (4)$	(6, $2k - 1$ ), $k \geq 6$	$\epsilon_1 \not\equiv 5 \ (8)$
(5, 5)	$\epsilon_2 \equiv 3\epsilon_1 + 6 \ (8)$	(6, $2k$ ), $k \geq 6$	$\epsilon_1, \epsilon_2 \not\equiv 5 \ (8)$ and $\epsilon_1 \not\equiv \epsilon_2 \Rightarrow \epsilon_1 \text{ or } \epsilon_2 \equiv 1 \ (8)$

**Remark 3.1.4.** For a primitive ternary  $\mathbb{Z}$ -lattice  $L$ , if  $L$  is of  $H$ -type at  $p$ , then

$$\lambda_p : \text{gen}(L)_S \longrightarrow \text{gen}(\lambda_p(L))_S$$

is bijective. Furthermore, if  $L$  is not of  $H$ -type at  $p$ , then we have

$$|\theta(O^+(\lambda_p(L_p)))| = \begin{cases} 4 \cdot |\theta(O^+(L_p))| & \text{if } p = 2, (\alpha, \beta) = (2, 4) \text{ and } \epsilon_1 \equiv \epsilon_2 \equiv 1 \ (4), \\ 2 \cdot |\theta(O^+(L_p))| & \text{otherwise.} \end{cases}$$

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Assume  $\lambda_p(\text{spn}(L)) = \text{spn}(M)$  with  $\lambda_p(L) = M$ . Let  $\text{spn}(M') \in \text{gen}(M)_S$ . Then there is a split rotation  $\Sigma \in J_V$  such that  $M' = \Sigma M$ . Since

$$\lambda_p(\Sigma L) = \Sigma \lambda_p(L) = \Sigma M = M',$$

we have  $\lambda_p(\text{spn}(\Sigma L)) = \text{spn}(M')$ . Note that  $\text{spn}(L') = \text{spn}(L'')$  if and only if  $\text{spn}(\Sigma L') = \text{spn}(\Sigma L'')$  for any  $L', L'' \in \text{gen}(L)$ . Therefore

$$\lambda_p : \text{gen}(L)_S \longrightarrow \text{gen}(\lambda_p(L))_S$$

is a four-to-one or two-to-one map if  $L$  is not of  $H$ -type at  $p$ . Note that  $\lambda_p$  could be a two-to-one map even if  $|\theta(O^+(\lambda_p(L_p)))| = 4 \cdot |\theta(O^+(L_p))|$ .

### 3.2 Generalization of Watson transformations

Let  $L$  be a positive definite non-classic integral ternary  $\mathbb{Z}$ -lattice. Assume  $p$  is an odd prime. If a unimodular component in the Jordan decomposition of  $L_p$  is anisotropic, then one may easily show that

$$r(pn, L) = r(pn, \Lambda_p(L)) \tag{3.2.1}$$

for every integer  $n$ . Therefore, we know that

$$r(n, \lambda_p(L)) = \begin{cases} r(pn, L) & \text{if } p\mathbb{Z}_p\text{-modular component of } L_p \text{ is non-zero,} \\ r(p^2n, L) & \text{otherwise.} \end{cases}$$

Furthermore, One may easily show that (3.2.1) still holds for  $p = 2$  unless

$$L_2 \simeq \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \perp \langle \alpha \rangle, \quad (\alpha \in \mathbb{Z}_2).$$

In the remaining of this section, we always assume that in a Jordan decomposition of  $L_p$ ,

$$\text{the } \frac{1}{2}\mathbb{Z}_p\text{-modular component is non-zero isotropic.} \tag{3.2.2}$$



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In this section we want to find similar results to (3.2.1) under the above assumption. For this, we generalize the Watson transformation variously. Since

$$\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \perp \langle \delta \rangle \simeq \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \perp \langle 5\delta \rangle \quad \text{over } \mathbb{Z}_2$$

for any  $\delta \in \mathbb{Z}_2^\times$ , any  $\mathbb{Z}$ -lattice  $L$  with above structure over  $\mathbb{Z}_2$  will also be considered when  $p = 2$ .

This section is a part of [13], we bring it here intactly.

**Definition 3.2.1.** Assume that  $p$  is odd. For  $\epsilon = 0$  or  $\pm 1$ , we define

$$S_p(\epsilon, L) = \left\{ x \in L \mid \left( \frac{Q(x)}{p} \right) = \epsilon \right\}.$$

We also define

$$S_2(0, L) = \{x \in L \mid Q(x) \equiv 0 \pmod{2}\}$$

and

$$S_2(*, L) = L - S_2(0, L).$$

Let  $\mathfrak{B} = \{x_1, x_2, x_3\}$  be a (ordered) basis of a ternary  $\mathbb{Z}$ -lattice  $L$  and  $p$  be a prime. We define a natural projection map

$$\phi_{\mathfrak{B}} : L - pL \rightarrow (L/pL)^* \rightarrow \mathbb{P}^2,$$

where  $\mathbb{P}^2$  is the 2-dimensional projective space over the finite field  $\mathbb{F}_p$ . The set  $\phi_{\mathfrak{B}}(S_p(\epsilon, L) - pL)$  is denoted by  $s_p^{\mathfrak{B}}(\epsilon, L)$  for any  $\epsilon \in \{0, 1, -1\}$  if  $p$  is odd and  $\epsilon \in \{0, *\}$  otherwise. If the basis  $\mathfrak{B}$  is obvious, we will omit it. For each element  $\mathbf{s} \in \mathbb{P}^2$ , we define a  $\mathbb{Z}$ -sublattice  $L_{\mathbf{s}} := \phi_{\mathfrak{B}}^{-1}(\mathbf{s}) \cup pL$  of  $L$ , and

$$\Omega_p(\epsilon, L) = \{L_{\mathbf{s}} \mid \mathbf{s} \in s_p^{\mathfrak{B}}(\epsilon, L)\}.$$

Note that if  $T : \mathfrak{B} \rightarrow \mathfrak{C}$  is the transition matrix between ordered bases, then one may easily show that  $T(s_p^{\mathfrak{B}}(\epsilon, L)) = s_p^{\mathfrak{C}}(\epsilon, L)$ . Hence the set  $\Omega_p(\epsilon, L)$  is independent of choices of the basis for  $L$ .

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**Lemma 3.2.2.** *Assume that a ternary  $\mathbb{Z}$ -lattice  $L$  and a prime  $p$  satisfies the condition (2.2). If  $4 \cdot dL_p \in \mathbb{Z}_p^\times$ , then*

$$|s_p(0, L)| = p + 1, \quad |s_p(\pm 1, L)| = \frac{p \left( p \pm \left( \frac{-dL}{p} \right) \right)}{2} \quad \text{and} \quad s_2(*, L) = 4$$

and

$$|s_p(0, L)| = 2p + 1, \quad |s_p(1, L)| = |s_p(-1, L)| = \frac{p(p-1)}{2} \quad \text{and} \quad s_2(*, L) = 2,$$

otherwise.

*Proof.* Since everything is trivial for  $p = 2$ , we assume that  $p$  is odd. For the unimodular case, see Theorem 1.3.2 of [15]. Assume that  $L_p$  is not unimodular. Fix an ordered basis  $\mathfrak{B} = \{x_1, x_2, x_3\}$  of  $L$  such that

$$(B(x_i, x_j)) \equiv \text{diag}(1, -1, p^{\text{ord}_p(dL)} \delta) \pmod{p^{\text{ord}_p(dL)+1}},$$

for some  $\delta \in \mathbb{Z} - p\mathbb{Z}$ . Note that such a basis always exists by Weak Approximation Theorem for Rotations. Assume  $x = a_1x_1 + a_2x_2 + a_3x_3 \in S_p(0, L)$ . Then  $a_1^2 \equiv a_2^2 \pmod{p}$ . Therefore

$$s_p^{\mathfrak{B}}(0, L) = \{(0, 0, 1), (1, \pm 1, d)\}, \quad \text{where } d \in \mathbb{F}_p.$$

The lemma follows from this. The case when  $\epsilon = \pm 1$  can be done in a similar manner.  $\square$

**Lemma 3.2.3.** *Under the same assumptions given above, assume that  $p$  is an odd prime. If  $\epsilon \neq 0$  or  $\epsilon = 0$  and  $L_p$  is unimodular, then every  $\mathbb{Z}$ -lattice  $M \in \Omega_p(\epsilon, L)$  is contained in one genus. Furthermore for the former case,*

$$M_q \simeq \begin{cases} \langle \delta, -p^2\delta, -p^2dL \rangle & \text{if } q = p, \\ L_q & \text{otherwise,} \end{cases}$$

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where  $\delta \in \mathbb{Z}_p^\times$  such that  $\left(\frac{\delta}{p}\right) = \epsilon$  and,

$$M_q \simeq \begin{cases} \langle p, -p, -p^2 dL \rangle & \text{if } q = p, \\ L_q & \text{otherwise,} \end{cases}$$

for the latter case. If  $L_p$  is not unimodular and  $\epsilon = 0$  then every  $\mathbb{Z}$ -lattice  $M \in \Omega_p(0, L)$  is exactly contained in two genera. More precisely

$$M_q \simeq \begin{cases} \langle p^2, -p^2, -dL \rangle \text{ or } \langle p, -p, -p^2 dL \rangle & \text{if } q = p, \\ L_q & \text{otherwise.} \end{cases}$$

*Proof.* Let  $L = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3$  and  $M \in \Omega_p(\epsilon, L)$ . Since  $pL \subset M$ , we may assume without loss of generality that

$$M = \mathbb{Z}(x_1 + b_2x_2 + b_3x_3) + \mathbb{Z}(px_2) + \mathbb{Z}(px_3).$$

First assume that  $\epsilon \neq 0$ . We may further assume that  $\left(\frac{Q(x_1 + b_2x_2 + b_3x_3)}{p}\right) = \epsilon$ . Since  $Q(x_1 + b_2x_2 + b_3x_3) \in \mathbb{Z}_p^\times$ ,

$$M_p \simeq \langle Q(x_1 + b_2x_2 + b_3x_3) \rangle \perp m_p$$

for some binary sublattice  $m_p$  of  $M_p$  whose scale is  $p^2\mathbb{Z}_p$ . The assertion follows from this. Assume that  $\epsilon = 0$  and  $L_p$  is unimodular. In this case we may assume that  $Q(x_1 + b_2x_2 + b_3x_3) \in p\mathbb{Z}_p$ . Then  $B(x_1 + b_2x_2 + b_3x_3, x_2)$  or  $B(x_1 + b_2x_2 + b_3x_3, x_3)$  is a unit in  $\mathbb{Z}_p$ , for  $L_p$  is unimodular. The assertion follows from this.

Finally assume that  $L_p$  is not unimodular and  $\epsilon = 0$ . In this case we may assume that the ordered basis  $\mathfrak{B} = \{x_1, x_2, x_3\}$  satisfies every condition in Lemma 3.2.2. Then by a direct computation we know  $L_{(0,0,1)} \in \Omega_p(0, L)$  satisfies the first local property and the others satisfy the second local property.  $\square$

**Lemma 3.2.4.** *Under the same assumptions given above, assume that  $p = 2$ .*

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Let  $M$  be a  $\mathbb{Z}$ -lattice in  $\Omega_2(\epsilon, L)$ . If  $-4 \cdot dL_2 = \delta \in \mathbb{Z}_2^\times$ , then

$$M_2 \simeq \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \langle 4\delta \rangle & \text{if } \epsilon = 0, \\ \langle 1, -1, 4\delta \rangle \quad \text{or} \quad \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \perp \langle \delta \rangle & \text{otherwise,} \end{cases}$$

and  $M_q \simeq L_q$  for any prime  $q \neq 2$ . If  $-4 \cdot dL_2 = \delta \in 2\mathbb{Z}_2$ , then

$$M_2 \simeq \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \langle 4\delta \rangle \quad \text{or} \quad \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \perp \langle \delta \rangle & \text{if } \epsilon = 0, \\ \langle 1, -1, 4\delta \rangle & \text{otherwise,} \end{cases}$$

and  $M_q \simeq L_q$  for any prime  $q \neq 2$ .

*Proof.* The proof is quite similar to the above.  $\square$

**Lemma 3.2.5.** Assume that a ternary  $\mathbb{Z}$ -lattice  $L$  and a prime  $p$  satisfies the condition (2.2). For any positive integer  $n$  such that  $\left(\frac{n}{p}\right) = \epsilon$ ,

$$r(n, L) = \sum_{M \in \Omega_p(\epsilon, L)} r(n, M) - (|s_p(\epsilon, L)| - 1)r(n, pL).$$

This equality also holds for  $p = 2$  if either  $\epsilon = 0$  and  $n$  is even or  $\epsilon = *$  and  $n$  is odd.

*Proof.* The lemma follows from the facts that

$$\{x \in S_p(\epsilon, L) - pL \mid Q(x) = n, \phi(x) = s\} = \{x \in L_s \mid Q(x) = n\} - R(n, pL),$$

and

$$L_s \cap L_t = pL \quad \text{if and only if} \quad s \neq t,$$

for any  $s, t \in \mathbb{P}^2$ .  $\square$

Under the same assumptions given above, one may easily show that  $dM = p^4 dL$  for any  $M \in \Omega_p(\epsilon, L)$ . Furthermore  $L/M \simeq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ .

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**Remark 3.2.6.** If a  $\frac{1}{2}\mathbb{Z}_p$ -modular component of  $L_p$  is zero or anisotropic, the above lemma implies the equation (3.2.1). So we may consider the above lemma as a natural generalization of Watson's transformation.

Let  $L$  and  $\ell$  be ternary  $\mathbb{Z}$ -lattices such that  $d\ell = p^4dL$ . We define

$$\tilde{R}(\ell, L) = \{\sigma : \ell \rightarrow L \mid L/\sigma(\ell) \simeq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}\} \quad \text{and} \quad \tilde{r}(\ell, L) = |\tilde{R}(\ell, L)|.$$

One may easily show that  $|\{M \in \Omega_p(\epsilon, L) \mid M \simeq \ell\}| = \tilde{r}(\ell, L)/o(\ell)$  for any  $\epsilon \in \{0, \pm 1\}$  or  $\epsilon \in \{0, *\}$ .

**Lemma 3.2.7.** *For any ternary  $\mathbb{Z}$ -lattices  $\ell$  and  $L$  such that  $d\ell = p^4dL$ , we have*

$$\tilde{r}(\ell, L) = r(p\ell^\#, L^\#) = r(pL, \ell).$$

*Proof.* Assume that  $T \in \tilde{R}(\ell, L)$ . Then  $T^t M_L T = M_\ell$  and  $pT^{-1}$  is an integral matrix. Since

$$(pT^{-1})M_L^{-1}(pT^{-1})^t = p^2M_\ell^{-1},$$

$(pT^{-1})^t \in R(p\ell^\#, L^\#)$ . Conversely if  $S^t M_L^{-1} S = p^2M_\ell^{-1}$ , then  $d(S) = \pm p$ . Hence  $pS^{-1}$  is an integral matrix and  $(pS^{-1})^t \in \tilde{R}(\ell, L)$ . This completes the proof.  $\square$

Assume that a ternary  $\mathbb{Z}$ -lattice  $L$  and a prime  $p$  satisfies the condition (3.2.2). In the remaining of this section, we further assume  $\text{ord}_p(4 \cdot dL) \geq 2$ . Let  $K = \lambda_p(L)$  and let

$$\text{gen}_p^K(L) = \{L' \in \text{gen}(L) : \lambda_p(L') \simeq K\}.$$

For any integer  $n$ , we also define

$$r(n, \text{gen}_p^K(L)) = \sum_{\substack{[L'] \in \text{gen}(L) \\ \lambda_p(L') \simeq K}} \frac{r(n, L')}{o(L')}.$$

In fact, every  $\mathbb{Z}$ -lattice in  $\text{gen}_p^K(L)$  is isometric to one of  $\mathbb{Z}$ -lattices in

$$\Gamma_p^L(\Lambda_p(L)) = \{M \subset K \mid M \in \text{gen}(L)\}.$$

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Furthermore, the isometry group  $O(K)$  acts on  $\Gamma_p^L(\Lambda_p(L))$ . Each orbit under this action consists of all isometric lattices in  $\Gamma_p^L(\Lambda_p(L))$ , and hence there are exactly  $\frac{o(K)}{o(L)}$  lattices that are isometric to  $L$  in  $\Gamma_p^L(\Lambda_p(L))$ . There are exactly  $p^2 + p + 1$  sublattices of  $K$  with index  $p$ . They are, in fact,

$$K_0 = \mathbb{Z}(px_1) + \mathbb{Z}x_2 + \mathbb{Z}x_3, \quad K_{1,u} = \mathbb{Z}(x_1 + ux_2) + \mathbb{Z}(px_2) + \mathbb{Z}x_3 \quad (0 \leq u \leq p-1)$$

and

$$K_{2,\alpha,\beta} = \mathbb{Z}(x_1 + \alpha x_3) + \mathbb{Z}(x_2 + \beta x_3) + \mathbb{Z}(px_3) \quad (0 \leq \alpha, \beta \leq p-1).$$

Among these sublattices of  $K$ , there are exactly  $\frac{p(p+1)}{2}$  lattices ( $p^2$  lattices) that are contained in the genus of  $L$  if  $\text{ord}_p(4 \cdot dL) = 2$  ( $\text{ord}_p(4 \cdot dL) \geq 3$ , respectively) (for details, see [5]).

**Proposition 3.2.8.** *Assume that  $\mathbb{Z}$ -lattices  $L$  and  $K$  and a prime  $p$  satisfies the above condition. Then for any integer  $n$  not divisible by  $p$ , we have*

$$r(n, \text{gen}_p^K(L)) = \begin{cases} \frac{p - \left(\frac{-ndK}{p}\right)}{2} \frac{r(n, K)}{o(K)} & \text{if } p \neq 2 \text{ and } \text{ord}_p(4 \cdot dL) = 2, \\ \frac{r(n, K) - r(n, \Lambda_1(K))}{o(K)} & \text{if } p = 2 \text{ and } \text{ord}_p(4 \cdot dL) = 2, \\ p \frac{r(n, K)}{o(K)} & \text{if } \text{ord}_p(4 \cdot dL) \geq 3, \end{cases}$$

where  $\Lambda_1(K) = \{x \in K : B(x, K) \subset \mathbb{Z}\}$  is a sublattice of  $K$ .

*Proof.* Since proofs are quite similar to each other, we only provide the proof of the first case. Assume that  $Q(x_1) = n$  for some  $x_1 \in K$ . We will count the number of lattices containing the vector  $x_1$  in  $\Gamma_p^L(\Lambda_p(L))$ . Note that for any vector  $y \in K$  and any integer  $d$  not divisible by  $p$ ,  $dy \in M$  if and only if  $y \in M$  for any  $M \in \Gamma_p^L(\Lambda_p(L))$ . Hence we may assume that  $x_1$  is a primitive vector in  $K$ . Then there is a basis  $\{x_1, x_2, x_3\}$  of  $K$  such that for some integer  $t$  not divisible by  $p$ ,

$$(B(x_i, x_j)) \equiv \text{diag}(n, n, t) \pmod{p}.$$

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Among all sublattices of  $K$  with index  $p$  that are contained in the genus of  $L$ , those  $\mathbb{Z}$ -lattices containing  $x_1$  are  $K_{2,0,\beta}$ , for any  $\beta$  satisfying  $\left(\frac{-n^2-n\beta^2 dK}{p}\right) = 1$ , and  $K_{1,0}$  only when  $\left(\frac{-ndK}{p}\right) = 1$ . Therefore one may easily show that the total number of such lattices is  $\frac{p - \left(\frac{-ndK}{p}\right)}{2}$ . The proposition follows from

$$\sum_{M \in \Gamma_p^L(\lambda_p(L))} r(n, M) = \sum_{[M] \in \text{gen}_p^K(L)} \frac{o(K)}{o(M)} r(n, M) = \frac{p - \left(\frac{-ndK}{p}\right)}{2} r(n, K).$$

This completes the proof.  $\square$

**Proposition 3.2.9.** *Under the same assumption given above, if  $n$  is divisible by  $p$ , then we have*

$$r(n, \text{gen}_p^K(L)) = \begin{cases} p \frac{r(n, K)}{o(K)} + \frac{p(p-1)}{2} \frac{r\left(\frac{n}{p^2}, K\right)}{o(K)} & \text{if } \text{ord}_p(4 \cdot dL) = 2, \\ p \frac{r(n, K)}{o(K)} + p^2 \frac{r\left(\frac{n}{p^2}, K\right)}{o(K)} - p \frac{r(n, \Lambda_p(K))}{o(K)} & \text{otherwise.} \end{cases}$$

*Proof.* First we define

$$R^*(n, K) = \{x \in K \mid Q(x) = n, \text{ } x \text{ is primitive as a vector in } K_p\},$$

$r^*(n, K) = |R^*(n, K)|$ , and  $r^\diamond(n, K) = r(n, K) - r^*(n, K)$ . Let  $x_1 \in K$  be a vector such that  $Q(x_1) = n$ . We will compute the number of lattices containing  $x_1$  in  $\Gamma_p^L(\Lambda_p(L))$ . By the similar reasoning to the above, we may assume that there is a primitive vector  $\tilde{x}_1 \in K$  and a nonnegative integer  $k$  such that  $x_1 = p^k \tilde{x}_1$ . If  $k > 0$ , then  $x_1$  is contained in all lattices in  $\Gamma_p^L(\Lambda_p(L))$ .

Assume that  $k = 0$ . If  $\text{ord}_p(4 \cdot dL) = 2$ , then there is a basis  $\{x_1, x_2, x_3\}$  of  $K$  such that

$$(B(x_i, x_j)) \equiv \begin{pmatrix} 0 & b & 0 \\ b & 0 & 0 \\ 0 & 0 & e \end{pmatrix} \pmod{p},$$

where  $2b$  and  $e$  are integers not divisible by  $p$ . Among all sublattices of  $K$

### CHAPTER 3. WATSON TRANSFORMATIONS

with index  $p$  that are contained in the genus of  $L$ , those  $\mathbb{Z}$ -lattices containing  $x_1$  are  $K_{2,0,\beta}$  for any  $\beta$ . Therefore if  $\text{ord}_p(4 \cdot dL) = 2$ , we have

$$\begin{aligned} \sum_{[M] \in \text{gen}_p^K(L)} \frac{o(K)}{o(M)} r(n, M) &= p \cdot r^*(n, K) + \frac{p(p+1)}{2} r^\diamond(n, K) \\ &= p \cdot r(n, K) + \frac{p(p-1)}{2} r\left(\frac{n}{p^2}, K\right). \end{aligned}$$

Suppose that  $\text{ord}_p(4 \cdot dL) \geq 3$ . If there is a vector  $y \in K$  such that  $2B(x_1, y) \not\equiv 0 \pmod{p}$ , then there are exactly  $p$  lattices in  $\Gamma_p^L(\Lambda_p(L))$  containing  $x_1$ . However if  $2B(x_1, K) \subset p\mathbb{Z}$ , then there does not exist a lattice in  $\Gamma_p^L(\Lambda_p(L))$  that contains  $x_1$ . Note that

$$|\{x \in R^*(n, K) \mid 2B(x, K) \subset p\mathbb{Z}\}| = r(n, \Lambda_p(K)) - r^\diamond(n, K).$$

Therefore we have

$$\sum_{[M] \in \text{gen}_p^K(L)} \frac{o(K)}{o(M)} r(n, M) = p(r(n, K) - r(n, \Lambda_p(K))) + p^2 \cdot r^\diamond(n, K).$$

This completes the proof. □



## Chapter 4

# Finite (multi-) graphs of ternary quadratic forms

In this chapter, we introduce a graph  $\mathfrak{G}_{L,p}(m)$  which is first defined in [13]. For most results in Section 4.1 and 4.2, one may also see [13].

### 4.1 Definition of the graph $\mathfrak{G}_{L,p}(m)$

Let  $V$  be a (positive definite) ternary quadratic space and let  $L$  be a (non-classic) ternary  $\mathbb{Z}$ -lattice on  $V$ . Let  $p$  be a prime such that

$$L_p \simeq \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \perp \langle \epsilon \rangle, \quad (4.1.1)$$

where  $\epsilon \in \mathbb{Z}_p^\times$ . For any nonnegative integer  $m$ , let  $\mathcal{G}_{L,p}(m)$  be a genus on  $W$  such that each  $\mathbb{Z}$ -lattice  $T \in \mathcal{G}_{L,p}(m)$  satisfies

$$T_p \simeq \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \perp \langle \epsilon p^m \rangle \quad \text{and} \quad T_q \simeq (L^p)^m_q \quad \text{for any } q \neq p.$$

Here  $W = V$  if  $m$  is even,  $W = V^p$  otherwise.

**Lemma 4.1.1.** *Let  $T \in \mathcal{G}_{L,p}(m)$  and  $S \in \mathcal{G}_{L,p}(m+1)$  be ternary  $\mathbb{Z}$ -lattices.*

## CHAPTER 4. FINITE (MULTI-) GRAPHS OF TERNARY FORMS

Then we have

$$\sum_{[N] \in \mathcal{G}_{L,p}(m+1)} \frac{\tilde{r}(N^p, T)}{o(N)} = \begin{cases} p+1 & \text{if } m = 0, \\ 2p & \text{otherwise} \end{cases}$$

and

$$\sum_{[M] \in \mathcal{G}_{L,p}(m)} \frac{r(M^p, S)}{o(M)} = 2.$$

*Proof.* Note that  $\sum_{[N] \in \mathcal{G}_{L,p}(m+1)} \frac{\tilde{r}(N^p, T)}{o(N)}$  is the number of sublattices  $X$  of  $T$  such that

$$T/X \simeq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \quad \text{and} \quad X^{\frac{1}{p}} \in \mathcal{G}_{L,p}(m+1).$$

Hence the first equality is a direct consequence of Lemmas 3.2.2, 3.2.3 and 3.2.4.

To prove the second equality, it suffices to show that there are exactly two sublattices of  $S$  with index  $p$  whose norm is  $p\mathbb{Z}$ . By Weak Approximation Theorem for Rotations, there exists a basis  $\{x_1, x_2, x_3\}$  for  $S$  such that

$$(B(x_i, x_j)) \equiv \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \perp \langle p^{m+1}\delta \rangle \pmod{p^{m+2}},$$

where  $\delta$  is an integer not divisible by  $p$ . Then for the following two sublattices defined by

$$\Gamma_{p,1}(S) = \mathbb{Z}px_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3, \quad \Gamma_{p,2}(S) = \mathbb{Z}x_1 + \mathbb{Z}px_2 + \mathbb{Z}x_3,$$

one may easily show that  $\Gamma_{p,i}(S)^{\frac{1}{p}} \in \mathcal{G}_{L,p}(m)$  for any  $i = 1, 2$ . Furthermore, norms of all the other sublattices of  $S$  with index  $p$  are not contained in  $p\mathbb{Z}$ . This completes the proof.  $\square$

Note that  $\Gamma_{p,i}(S)$  for  $i = 1, 2$  depends on the choices of basis for  $S$ . However, the set  $\{\Gamma_{p,1}(S), \Gamma_{p,2}(S)\}$  of sublattices of  $S$  is independent of the choices of basis for  $S$ . In fact, they are unique sublattices of  $S$  with index  $p$  whose norm is  $p\mathbb{Z}$ . We say that a  $\mathbb{Z}$ -lattice  $T$  is a  $\Gamma_p$ -descendant of  $S$  if

## CHAPTER 4. FINITE (MULTI-) GRAPHS OF TERNARY FORMS

$T \simeq \Gamma_{p,i}^{\frac{1}{p}}(S)$  for some  $i = 1, 2$ .

**Lemma 4.1.2.** *Let  $p, q$  be distinct primes and let  $S \in \mathcal{G}_{L,p}(m+1)$  for some nonnegative integer  $m$ .*

- (a) *If  $T$  is a  $\Gamma_p$ -descendant of  $S$ , then  $\lambda_q(T)$  is a  $\Gamma_p$ -descendant of  $\lambda_q(S)$ .*
- (b) *Assume that  $S \in \mathcal{G}_{L',q}(m'+1)$  for some nonnegative integer  $m'$ . Then any  $\Gamma_q$ -descendant of a  $\Gamma_p$ -descendant of  $S$  is a  $\Gamma_p$ -descendant of some  $\Gamma_q$ -descendant of  $S$ .*

*Proof.* If  $p, q$  are distinct primes, then  $(\Gamma_{p,i}(S))_q = S_q$  and  $(\Lambda_p(S))_q = S_q$ . The lemma follows directly from this.  $\square$

Now we define a multi-graph  $\mathfrak{G}_{L,p}(m)$  as follows: the set of vertices in  $\mathfrak{G}_{L,p}(m)$  is the set of equivalence classes in  $\mathcal{G}_{L,p}(m)$ , say,

$$\{[T_1], [T_2], \dots, [T_h]\}.$$

The set of edges is exactly the set of equivalence classes in  $\mathcal{G}_{L,p}(m+1)$ , say,

$$\{[S_1], [S_2], \dots, [S_k]\}.$$

For each equivalence class  $[S_w] \in \mathcal{G}_{L,p}(m+1)$ , two vertices contained in the edge named by  $[S_w]$  are defined by

$$[\Gamma_{p,1}(S_w)^{\frac{1}{p}}] \text{ and } [\Gamma_{p,2}(S_w)^{\frac{1}{p}}],$$

where the lattice  $\Gamma_{p,i}(S_w)^{\frac{1}{p}}$  that is defined in Lemma 4.1.1 is contained in  $\mathcal{G}_{L,p}(m)$ . Note that the graph  $\mathfrak{G}_{L,p}(m)$  is, in general, a multi-graph that might have a loop. We define an  $h \times k$  integer matrix  $\mathfrak{M}_{L,p}(m) = (m_{ij})$  as follows:

$$m_{ij} = \begin{cases} 2 & \text{if } [S_j] \text{ is a loop of the vertex } [T_i], \\ 1 & \text{if } [S_j] \text{ is not a loop of the vertex } [T_i], \text{ though it contains } [T_i], \\ 0 & \text{otherwise.} \end{cases}$$

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Therefore  $\mathfrak{M}_{L,p}(m)$  is the incidence matrix of  $\mathfrak{G}_{L,p}(m)$  if the graph  $\mathfrak{G}_{L,p}(m)$  is simple.

### 4.2 Connected components

Let  $p$  be a prime and  $L$  be a non-classic ternary  $\mathbb{Z}$ -lattice on  $V$  satisfying the equation (4.1.1).

For any  $\mathbb{Z}$ -lattice  $T \in \mathcal{G}_{L,p}(m)$ , we define

$$\Phi_p(T) = \{S \in \mathcal{G}_{L,p}(m+1) : \Gamma_{p,i}(S)^{\frac{1}{p}} = T \text{ for some } i = 1, 2\}$$

and

$$\Psi_p(T) = \{M \in \mathcal{G}_{L,p}(m+2) : \lambda_p(M) = T\}.$$

Then Lemma 4.1.1 implies that  $|\Phi_p(T)| = p+1$  if  $m = 0$ ,  $|\Phi_p(T)| = 2p$  otherwise.

**Lemma 4.2.1.** *Let  $T \in \mathcal{G}_{L,p}(0)$  and  $S, S' \in \Phi_p(T)$  ( $S \neq S'$ ) be ternary  $\mathbb{Z}$ -lattices on  $V$  and  $V^p$ , respectively. Then there is a unique  $\mathbb{Z}$ -lattice  $M$  in  $\Psi_p(T)$  such that  $\{\Gamma_{p,1}(M)^{\frac{1}{p}}, \Gamma_{p,2}(M)^{\frac{1}{p}}\} = \{S, S'\}$ .*

*Proof.* For any  $S, S' \in \Phi_p(T)$ , we have  $pS \subset S'$ . Furthermore since  $S \neq S'$  and  $\text{ord}_p(4 \cdot dS) = 1$ ,  $S'/pS \simeq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$ . Therefore, there is a basis  $x_1, x_2, x_3$  for  $S'$  such that

$$S' = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3, \quad pS = \mathbb{Z}x_1 + \mathbb{Z}px_2 + \mathbb{Z}p^2x_3$$

and

$$(B(x_i, x_j)) = \begin{pmatrix} p^2a & pb & d \\ pb & pc & e \\ d & e & f \end{pmatrix},$$

where  $a, c, f \in \mathbb{Z}$ ,  $b, d, e \in \frac{1}{2}\mathbb{Z}$  and  $p \nmid 2d$ . Define a  $\mathbb{Z}$ -lattice

$$M = \left( \mathbb{Z} \begin{pmatrix} x_1 \\ p \end{pmatrix} + \mathbb{Z}x_2 + \mathbb{Z}x_3 \right)^p \in \mathcal{G}_{L,p}(2).$$

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Then one may easily show that  $\lambda_p(M) = T$  and

$$\{\Gamma_{p,1}(M)^{\frac{1}{p}}, \Gamma_{p,2}(M)^{\frac{1}{p}}\} = \{S, S'\}.$$

As pointed out earlier, the number of ternary  $\mathbb{Z}$ -lattices  $M' \in \mathcal{G}_{L,p}(2)$  such that  $\lambda_p(M') = T$  for any  $T \in \mathcal{G}_{L,p}(0)$  is  $\frac{p(p+1)}{2}$ . Furthermore for any such a  $\mathbb{Z}$ -lattice  $M'$ , we have  $\Gamma_{p,i}(M')^{\frac{1}{p}} \in \Phi_p(T)$  for any  $i = 1, 2$  and  $|\Phi_p(T)| = p+1$ . Now the uniqueness of  $M$  follows from this observation.  $\square$

The above lemma says that if  $T \in \mathcal{G}_{L,p}(0)$ , then there is always an edge containing  $[S]$  and  $[S']$  for any  $S, S' \in \Phi_p(T)$ . However this is not true in general if  $T \in \mathcal{G}_{L,p}(m)$  for a positive integer  $m$ .

**Lemma 4.2.2.** *For a positive integer  $m$ , let  $T \in \mathcal{G}_{L,p}(m)$  and  $S, S' \in \Phi_p(T)$  be ternary  $\mathbb{Z}$ -lattices on  $V$  and  $V^p$ , respectively. If*

$$\lambda_p(S) = \Gamma_{p,1}(T)^{\frac{1}{p}} \quad \text{and} \quad \lambda_p(S') = \Gamma_{p,2}(T)^{\frac{1}{p}},$$

*then there is a unique  $\mathbb{Z}$ -lattice  $M \in \Psi_p(T)$  such that*

$$\{\Gamma_{p,1}(M)^{\frac{1}{p}}, \Gamma_{p,2}(M)^{\frac{1}{p}}\} = \{S, S'\}.$$

*Proof.* By Weak Approximation Theorem for Rotations, there is a basis  $x_1, x_2, x_3$  for  $T$  such that

$$(B(x_i, x_j)) \equiv \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \perp \langle p^m \delta \rangle \pmod{p^{m+1}},$$

where  $\delta$  is an integer not divisible by  $p$ . We may assume that

$$\Gamma_{p,1}(T)^{\frac{1}{p}} = (\mathbb{Z}px_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3)^{\frac{1}{p}}, \quad \Gamma_{p,2}(T)^{\frac{1}{p}} = (\mathbb{Z}x_1 + \mathbb{Z}px_2 + \mathbb{Z}x_3)^{\frac{1}{p}}.$$

One may easily check that

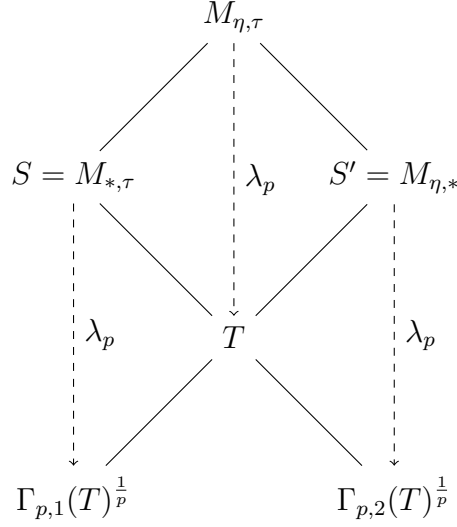
$$\begin{aligned} \Phi_p(T) = & \{M_{*,\beta} = (\mathbb{Z}px_1 + \mathbb{Z}(x_2 + \beta x_3) + \mathbb{Z}px_3)^{\frac{1}{p}} : 0 \leq \beta \leq p-1\} \\ & \cup \{M_{\alpha,*} = (\mathbb{Z}(x_1 + \alpha x_3) + \mathbb{Z}px_2 + \mathbb{Z}px_3)^{\frac{1}{p}} : 0 \leq \alpha \leq p-1\} \end{aligned}$$

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and

$$\Psi_p(T) = \{M_{\alpha,\beta} = \mathbb{Z}(x_1 + \alpha x_3) + \mathbb{Z}(x_2 + \beta x_3) + \mathbb{Z}px_3 : 0 \leq \alpha, \beta \leq p-1\}.$$

Since  $\lambda_p(M_{*,\beta}) = \Gamma_{p,1}(T)^{\frac{1}{p}}$  and  $\lambda_p(M_{\alpha,*}) = \Gamma_{p,2}(T)^{\frac{1}{p}}$  for any  $0 \leq \alpha, \beta \leq p-1$ , there are  $\tau, \eta$  such that  $S = M_{*,\tau}$  and  $S' = M_{\eta,*}$ .



### 3.1 $\lambda_p$ -transformation and connectivity(different vertices)

Now, one may easily check that  $M_{\eta,\tau}$  is the unique lattice in  $\Psi_p(T)$  satisfying

$$\{\Gamma_{p,1}(M_{\eta,\tau})^{\frac{1}{p}}, \Gamma_{p,2}(M_{\eta,\tau})^{\frac{1}{p}}\} = \{M_{*,\tau}, M_{\eta,*}\}.$$

This completes the proof. □

**Lemma 4.2.3.** *For an integer  $m \geq 2$ , let  $M_1, M_2 \in \mathcal{G}_{L,p}(m)$  be distinct  $\mathbb{Z}$ -lattices such that  $\lambda_p(M_1) = \lambda_p(M_2) = T$ . Then there is a path from  $[M_1]$  to  $[M_2]$  of length 4.*

*Proof.* Note that if  $\{\Gamma_{p,1}(M_1), \Gamma_{p,2}(M_1)\} = \{\Gamma_{p,1}(M_2), \Gamma_{p,2}(M_2)\}$ , then we know that  $M_1 = M_2$ . Hence, without loss of generality, we may assume that  $S_1 = \Gamma_{p,1}(M_1)^{\frac{1}{p}}$  is different from  $S_2 = \Gamma_{p,2}(M_2)^{\frac{1}{p}}$ . If  $m \geq 3$ , then

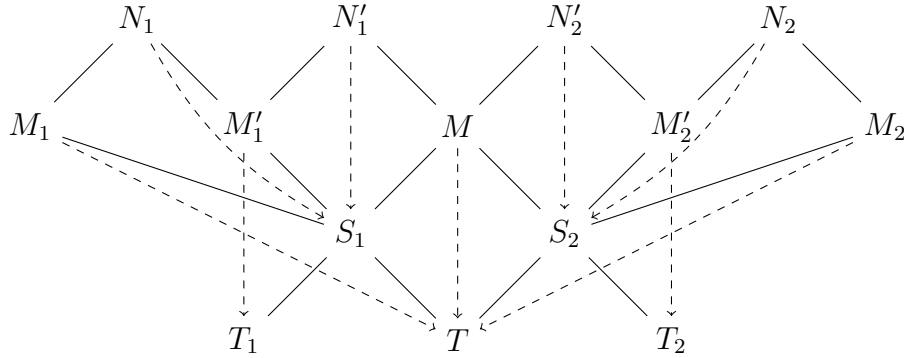
$$\{\lambda_p(\Gamma_{p,1}(M_i)^{\frac{1}{p}}), \lambda_p(\Gamma_{p,2}(M_i)^{\frac{1}{p}})\} = \{\Gamma_{p,1}(T)^{\frac{1}{p}}, \Gamma_{p,2}(T)^{\frac{1}{p}}\}$$

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for any  $i = 1, 2$ . Hence we further assume that  $\lambda_p(S_1) \neq \lambda_p(S_2)$ . Then by Lemmas 4.2.1 and 4.2.2, there is a  $\mathbb{Z}$ -lattice  $M \in \mathcal{G}_{L,p}(m)$  such that  $\lambda_p(M) = T$  and  $\{\Gamma_{p,1}(M)^{\frac{1}{p}}, \Gamma_{p,2}(M)^{\frac{1}{p}}\} = \{S_1, S_2\}$ . We define  $\mathbb{Z}$ -lattices  $T_1$  and  $T_2$  satisfying

$$\{\Gamma_{p,1}(S_1)^{\frac{1}{p}}, \Gamma_{p,2}(S_1)^{\frac{1}{p}}\} = \{T, T_1\} \quad \text{and} \quad \{\Gamma_{p,1}(S_2)^{\frac{1}{p}}, \Gamma_{p,2}(S_2)^{\frac{1}{p}}\} = \{T, T_2\}.$$

Let  $M'_i \in \mathcal{G}_{L,p}(m)$  be a  $\mathbb{Z}$ -lattice in  $\Phi_p(S_i)$  such that  $\lambda_p(M'_i) = T_i$  for any  $i = 1, 2$ . Then by Lemma 4.2.2, there are  $\mathbb{Z}$ -lattices  $N_1, N_2, N'_1, N'_2$  such that two vertices  $[M_i]$  and  $[M'_i]$  are connected by the edge  $[N_i]$ , and two vertices  $[M]$  and  $[M'_i]$  are connected by the edge  $[N'_i]$  for  $i = 1, 2$ . Therefore two vertices  $[M_1]$  and  $[M_2]$  are connected by a path of length 4 (see Figure 3.2).



### 3.2 $\lambda_p$ -transformation and connectivity(same vertex)

The Lemma follows from this. □

**Lemma 4.2.4.** *For an integer  $m \geq 2$ , let  $[M], [M']$  be vertices of the graph  $\mathfrak{G}_{L,p}(m)$ . Then there is a path from  $[M]$  to  $[M']$  of length  $e([M], [M'])$  in  $\mathfrak{G}_{L,p}(m)$  if and only if there is a path from  $[\lambda_p(M)]$  to  $[\lambda_p(M')]$  of length  $e([\lambda_p(M)], [\lambda_p(M')])$  in  $\mathfrak{G}_{L,p}(m-2)$ . Furthermore, in both cases, there is a path satisfying*

$$e([M], [M']) \equiv e([\lambda_p(M)], [\lambda_p(M')]) \pmod{2}.$$

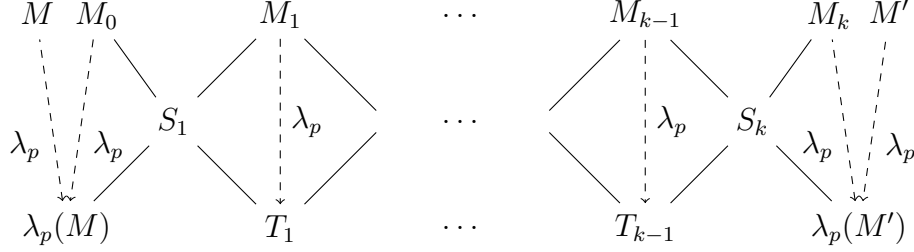
*Proof.* Note that “only if” part is trivial. Assume that  $[\lambda_p(M)]$  and  $[\lambda_p(M')]$

## CHAPTER 4. FINITE (MULTI-) GRAPHS OF TERNARY FORMS

are connected by a path with edges  $[S_1], [S_2], \dots, [S_k]$  as in Figure 3.3, where

$$\{\Gamma_{p,1}(S_i)^{\frac{1}{p}}, \Gamma_{p,2}(S_i)^{\frac{1}{p}}\} = \{T_{i-1}, T_i\}$$

for any  $i = 2, 3, \dots, k-1$ .



### 3.3 $\lambda_p$ -transformation and connectivity (general case)

Then for any  $i = 0, 1, \dots, k$ , there are ternary  $\mathbb{Z}$ -lattices  $M_i$  such that

$$M_0 \in \Psi_p(\lambda_p(M)) \cap \Phi_p(S_1), \quad M_k \in \Psi_p(\lambda_p(M')) \cap \Phi_p(S_k),$$

and

$$M_j \in \Psi_p(T_j) \cap \Phi_p(S_j) \cap \Phi_p(S_{j+1})$$

for any  $j = 1, 2, \dots, k-1$ . Now by Lemma 4.2.2, there are  $\mathbb{Z}$ -lattices  $N_i$  such that

$$\{\Gamma_{p,1}(N_i)^{\frac{1}{p}}, \Gamma_{p,2}(N_i)^{\frac{1}{p}}\} = \{M_{i-1}, M_i\} \quad \text{and} \quad \lambda_p(N_i) = S_i$$

for any  $i = 1, 2, \dots, k$ . Since both  $[M], [M_0]$  and  $[M_k], [M']$  are connected by a path of length 4 by Lemma 4.2.3,  $[M]$  and  $[M']$  are connected by a path of length  $k+8$ .  $\square$

We investigate the graph  $\mathfrak{G}_{L,p}(0)$  in more detail. Let  $T \in \mathcal{G}_{L,p}(0)$  be a  $\mathbb{Z}$ -lattice. Note that the graph  $Z(T, p)$  constructed in [18] is slightly different from our graph (see also [1]). In fact, the graph  $Z(T, p)$  is a tree having infinitely many vertices. However our graph is finite and might have a loop. Two vertices  $[T_i], [T_j] \in \mathfrak{G}_{L,p}(0)$  are connected by an edge if and only if there are  $\mathbb{Z}$ -lattices  $T'_i \in [T_i]$  and  $T'_j \in [T_j]$  such that  $T'_i$  and  $T'_j$  are connected by an edge in the graph  $Z(T, p)$ . If two lattices  $T_i, T_j \in \mathcal{G}_{L,p}(0)$  are spinor equivalent, then both  $[T_i]$  and  $[T_j]$  are contained in the same connected component.



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Moreover, each connected component of  $\mathfrak{G}_{L,p}(0)$  contains at most two spinor genera, and it contains only one spinor genus if and only if  $\mathbf{j}(p) \in P_D J_{\mathbb{Q}}^T$ , where  $D$  is the set of positive rational numbers and

$$\mathbf{j}(p) = (j_q) \in J_{\mathbb{Q}} \quad \text{such that } j_p = p \text{ and } j_q = 1 \text{ for any prime } q \neq p.$$

We say that  $\mathfrak{G}_{L,p}(0)$  is of  $O$ -type if each connected component of  $\mathfrak{G}_{L,p}(0)$  contains only one spinor genus, and it is of  $E$ -type otherwise. If  $\mathfrak{G}_{L,p}(0)$  is of  $E$ -type, then adjacent classes are contained in different spinor genera (for details, see [1]), that is, each connect component of the graph  $\mathfrak{G}_{L,p}(0)$  is a bipartite graph.

Assume that

$$\mathcal{G}_{L,p}(0) = \{[T_1], [T_2], \dots, [T_h]\} \quad \text{and} \quad \mathcal{G}_{L,p}(1) = \{[S_1], [S_2], \dots, [S_k]\} \quad (4.2.1)$$

are *ordered* sets of equivalence classes in each genus. We define

$$\mathfrak{M} = \left( \frac{r(T_i^p, S_j)}{o(T_i)} \right) \in M_{h,k}(\mathbb{Z}) \quad \text{and} \quad \mathfrak{N} = \mathfrak{N}_{L,p}(0) = \left( \frac{r(T_i^p, S_j)}{o(S_j)} \right) \in M_{h,k}(\mathbb{Z}).$$

In fact,  $\mathfrak{M}$  equals to  $\mathfrak{M}_{L,p}(0)$ , which is defined earlier. There is a nice relation between  $\mathfrak{M}$ ,  $\mathfrak{N}$  and the *Eichler's Anzahlmatrix*  $\pi_p(T)$  defined in [8].

**Definition 4.2.5.** Under the assumptions given above, the matrix

$$\pi_p(T) = \left( \frac{r(pT_i, T_j)}{o(T_i)} - \delta_{ij} \right) \quad (1 \leq i, j \leq h)$$

is called the Eichler's Anzahlmatrix of  $T$  at  $p$ .

Note that  $\pi_p(T)$  is independent of the choice of the lattice  $T \in \mathcal{G}_{L,p}(0)$ .

**Lemma 4.2.6.** *For any  $\mathbb{Z}$ -lattices  $T \in \mathcal{G}_{L,p}(0)$  and  $S \in \mathcal{G}_{L,p}(1)$ , we have  $r(S^p, T) = r(T^p, S)$ .*

*Proof.* First we show that  $\tilde{R}(S^p, T) = R(S^p, T)$ . Suppose that there is a  $\sigma \in R(S^p, T)$  such that  $T/\sigma(S^p) \simeq \mathbb{Z}/p^2\mathbb{Z}$ . Then there is a basis for  $T$  such

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that

$$T = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3 \quad \text{and} \quad \sigma(S^p) = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}(p^2x_3).$$

Since  $\mathfrak{n}(\sigma(S^p)) \subset p\mathbb{Z}$ , we have

$$Q(x_1) \equiv Q(x_2) \equiv 2B(x_1, x_2) \equiv 0 \pmod{p}.$$

This is a contradiction to the fact that  $4 \cdot dT$  is not divisible by  $p$ . Therefore the lemma follows from Lemma 3.2.7.  $\square$

For  $\mathbb{Z}$ -lattices  $X_1, X_2, Y_1$  and  $Y_2$ , we write  $(X_1, X_2) \simeq (Y_1, Y_2)$  if  $X_1 \simeq Y_1$  and  $X_2 \simeq Y_2$ , or  $X_1 \simeq Y_2$  and  $X_2 \simeq Y_1$ .

**Proposition 4.2.7.** *Under the notations and assumptions given above, we have*

$$\pi_p(T) + (p+1)I = \mathfrak{M} \cdot \mathfrak{N}^t.$$

*Proof.* Let  $\mathfrak{U}_{ij}$  be the set of sublattices  $X$  of  $T_j$  such that

$$X \simeq pT_i \quad \text{and} \quad T_j/X \not\simeq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z},$$

and let  $\mathfrak{V}_{ij}$  be the set of sublattices  $Y$  of  $T_j$  such that

$$Y^{\frac{1}{p}} \in \mathcal{G}_{L,p}(1) \quad \text{and} \quad \left( \Gamma_{p,1}(Y^{\frac{1}{p}}), \Gamma_{p,2}(Y^{\frac{1}{p}}) \right) \simeq (T_i^p, T_j^p),$$

where  $\Gamma_{p,i}(Y^{\frac{1}{p}})$  is a sublattice of  $Y^{\frac{1}{p}}$  with index  $p$  defined in Lemma 4.1.1. Note that  $\pi_p(T)_{ij} = |\mathfrak{U}_{ij}|$ . Now we define a map  $\Phi : \mathfrak{U}_{ij} \mapsto \mathfrak{V}_{ij}$  as follows. Assume that  $X \in \mathfrak{U}_{ij}$ . Then one may easily show that  $T_j/X \simeq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$ . Hence there is a basis  $x_1, x_2, x_3$  for  $T_j$  such that

$$T_j = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3 \quad \text{and} \quad X = \mathbb{Z}x_1 + \mathbb{Z}(px_2) + \mathbb{Z}(p^2x_3).$$

Since the integer  $4 \cdot d(T_j)$  is not divisible by  $p$  and  $Q(x_1) \equiv 0 \pmod{p^2}$ ,  $2B(x_1, x_2) \equiv 0 \pmod{p}$ , neither  $Q(x_2)$  nor  $2B(x_1, x_3)$  is divisible by  $p$ . Define  $\Phi(X) := Y = \mathbb{Z}x_1 + \mathbb{Z}(px_2) + \mathbb{Z}(px_3)$ . Clearly,  $Y = \Lambda_p(T_j \cap \frac{1}{p}X)$ . Hence

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it is independent of the choice of basis for  $T_j$ . Furthermore one may easily check that  $\Phi(X) = Y \in \mathfrak{V}_{ij}$ . Conversely, there are exactly two sublattices of  $Y^{\frac{1}{p}}$  with index  $p$  whose norm is contained in  $p\mathbb{Z}$ , and one of them is equal to  $T_j^p$ . If we define the other one, as a sublattice of  $Y$ , by  $\Psi(Y)$ , then  $\Phi \circ \Psi = \Psi \circ \Phi = Id$ . Therefore  $\pi_p(T)_{ij} = |\mathfrak{V}_{ij}|$ . Now from the definition,

$$|\mathfrak{V}_{ij}| = \sum_{w=1}^k \frac{r(S_w^p, T_j)}{o(S_w)} \eta_w,$$

where

$$\eta_w = \begin{cases} 1 & \text{if } (\Gamma_{p,1}(S_w), \Gamma_{p,2}(S_w)) \simeq (T_j^p, T_i^p), \\ 0 & \text{otherwise.} \end{cases}$$

Since  $r(T_j^p, S_w) = r(S_w^p, T_j)$  by Lemma 4.2.6, each  $|\mathfrak{V}_{ij}|$  equals to

$$\sum_{w=1}^k \frac{r(S_w^p, T_j)}{o(S_w)} \left( \frac{r(T_i^p, S_w)}{o(T_i)} - \delta_{ij} \right) = \begin{cases} \sum_{w=1}^k \mathfrak{M}_{iw}(\mathfrak{N}^t)_{wj} & \text{if } i \neq j, \\ \sum_{w=1}^k \mathfrak{M}_{iw}(\mathfrak{N}^t)_{wj} - (p+1) & \text{if } i = j, \end{cases}$$

by Lemma 4.1.1. The proposition follows from this.  $\square$

The following theorem states that the rank of  $\mathfrak{M}_{L,p}(0) = \mathfrak{M}$  is related with some properties of the graph  $\mathfrak{G}_{L,p}(0)$ .

**Theorem 4.2.8.** *The followings are all equivalent:*

- (1)  $\mathfrak{G}_{L,p}(0)$  is of  $O$ -type;
- (2)  $\text{rank}(\mathfrak{M}) = h$ ;
- (3)  $\pi_p(T)$  does not have an eigenvalue  $-(p+1)$ ;
- (4)  $g^+(\mathcal{G}_{L,p}(0)) = g^+(\mathcal{G}_{L,p}(1))$ .

Furthermore, if  $\mathfrak{G}_{L,p}(0)$  is of  $E$ -type, then  $g^+(\mathcal{G}_{L,p}(0)) = 2g^+(\mathcal{G}_{L,p}(1))$ , where  $g^+(\mathcal{G}_{L,p}(0))$  is the number of spinor genera in  $\mathcal{G}_{L,p}(0)$ .

*Proof.* (1)  $\Leftrightarrow$  (2): Assume that  $\mathfrak{G}_{L,p}(0)$  is of  $O$ -type. Without loss of generality, we may assume that  $\mathfrak{G}_{L,p}(0)$  is connected, that is, every  $\mathbb{Z}$ -lattice

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in  $\mathfrak{G}_{L,p}(0)$  is spinor equivalent. It is well known that the rank of an incidence matrix of a connected graph  $G(V, E)$  over  $\mathbb{F}_2$  is  $|V| - 1$ . Furthermore if the graph  $G$  contains an odd cycle, then the rank of the incidence matrix of  $G$  over  $\mathbb{Q}$  is equal to the number of vertices. Hence it suffices to show that the graph  $\mathfrak{G}_{L,p}(0)$  contains an odd cycle, even though it might contain a loop. Assume that  $[T_1]$  and  $[T_2]$  be adjacent vertices in  $\mathfrak{G}_{L,p}(0)$ . Since they are spinor equivalent, there is an isometry  $\sigma \in O(V)$  and  $\Sigma = (\Sigma_p) \in J'_V$  such that  $T_1 = \sigma \Sigma(T_2)$ , where  $V = \mathbb{Q} \otimes T_1$ . Let  $\Phi = \{q \in P - \{p\} \mid (\sigma^{-1}(T_1))_q = (T_2)_q\}$  and  $\Psi = P - (\Phi \cup \{p\})$ , where  $P$  is the set of all primes. Now by Strong Approximation Theorem for Rotations, for any  $\epsilon > 0$ , there is a rotation  $\tau \in O'(V)$  such that

$$\|\tau - \Sigma_q\|_q < \epsilon \quad \text{for any } q \in \Psi \quad \text{and} \quad \|\tau\|_q = 1 \quad \text{for any } q \in \Phi.$$

Therefore we have

$$\sigma^{-1}(T_1)_q = \tau(T_2)_q \quad \text{for any } q \neq p \quad \text{and} \quad \Sigma_p \circ \tau^{-1}(\tau(T_2)_p) = \sigma^{-1}(T_1)_p,$$

where  $\Sigma_p \circ \tau^{-1} \in O'(V_p)$ . Consequently, there is an even integer  $n$  and a basis  $\{x_1, x_2, x_3\}$  for  $\tau(T_2)$  such that

$$\tau(T_2) = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3 \quad \text{and} \quad \sigma^{-1}(T_1) = \mathbb{Z}(p^n x_1) + \mathbb{Z}(p^{-n} x_2) + \mathbb{Z}x_3,$$

by Lemma 4.2 of [1]. This implies that there is a path from  $[T_1]$  to  $[T_2]$  with even edges, and hence the graph  $\mathfrak{G}_{L,p}(0)$  contains an odd cycle.

Assume that  $\mathfrak{G}_{L,p}(0)$  is of  $E$ -type. Since any two adjacent vertices are contained in different spinor genera in this case, it is a bipartite (multi-) graph. Therefore the rank of the matrix  $\mathfrak{M}_{L,p}(0)$  is  $h - 1$ .

**(2)  $\Leftrightarrow$  (3) :** Note that  $\text{rank}(\mathfrak{M}) = \text{rank}(\mathfrak{M}\mathfrak{M}^t)$ . Hence the assertion follows directly from Proposition 4.2.7.

**(1)  $\Leftrightarrow$  (4) :** Note that  $g^+(\mathcal{L}) = [J_{\mathbb{Q}} : P_D J_{\mathbb{Q}}^{\mathcal{L}}]$  for any genus  $\mathcal{L}$  with rank greater than 2. Since

$$P_D J_{\mathbb{Q}}^{\mathcal{G}_{L,p}(1)} = P_D J_{\mathbb{Q}}^{\mathcal{G}_{L,p}(0)} \cup \mathbf{j}(p) \cdot P_D J_{\mathbb{Q}}^{\mathcal{G}_{L,p}(0)},$$

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$g^+(\mathcal{G}_{L,p}(1)) = g^+(\mathcal{G}_{L,p}(0))$  if and only if  $\mathbf{j}(p) \in P_D J_{\mathbb{Q}}^{\mathcal{G}_{L,p}(0)}$ , that is,  $\mathfrak{S}_{L,p}(0)$  is of  $O$ -type. Furthermore if  $\mathfrak{S}_{L,p}(0)$  is of  $E$ -type, then  $g^+(\mathcal{G}_{L,p}(0)) = 2g^+(\mathcal{G}_{L,p}(1))$ .  $\square$

Now, we consider the general case. For any positive integer  $m$ , we say that a graph  $\mathfrak{S}_{L,p}(m)$  is of  $E$ -type if  $m$  is even and  $\mathfrak{S}_{L,p}(0)$  is of  $E$ -type, and  $O$ -type otherwise.

Assume that  $\mathfrak{S}_{L,p}(m)$  is of  $E$ -type and  $M \in \mathcal{G}_{L,p}(m)$ . Since the map

$$\lambda_p^{\frac{m}{2}} : \text{spn}(K) \rightarrow \text{spn}(\lambda_p^{\frac{m}{2}}(K))$$

is surjective for any  $K \in \mathcal{G}_{L,p}(m)$ , there is a  $\mathbb{Z}$ -lattice  $M' \in \mathcal{G}_{L,p}(m)$  such that  $M' \notin \text{spn}(M)$  and  $[M']$  is connected to  $[M]$  by a path by Lemma 4.2.4. Furthermore, since  $g^+(\mathcal{G}_{L,p}(m)) = g^+(\mathcal{G}_{L,p}(0))$  for any even  $m$ , every  $\mathbb{Z}$ -lattice  $M'$  satisfying the above condition forms a single spinor genus. From the existence of such a  $\mathbb{Z}$ -lattice  $[M']$ , we may define

$$\text{Cspn}(M) = \begin{cases} \text{spn}(M) & \text{if } \mathfrak{S}_{L,p}(m) \text{ is of } O\text{-type,} \\ \text{spn}(M) \cup \text{spn}(M') & \text{otherwise,} \end{cases}$$

**Lemma 4.2.9.** *For a  $\mathbb{Z}$ -lattice  $M \in \mathcal{G}_{L,p}(m)$ , the set of all vertices in the connected component of  $\mathfrak{S}_{L,p}(m)$  containing  $[M]$  is the set of equivalence classes in  $\text{Cspn}(M)$ .*

*Proof.* First, we prove the case when  $m = 1$ . Assume that  $M' \in \text{spn}(M)$ . Then there are  $\sigma \in P_V$  and  $\Sigma \in J'_V$  such that  $M' = \sigma \Sigma M$  (see [17]). Since  $\Gamma_{p,i}(M)$ 's are the only sublattices of  $M$  with index  $p$  whose norm is  $p\mathbb{Z}$ , we have

$$\{\sigma \Sigma(\Gamma_{p,1}(M)^{\frac{1}{p}}), \sigma \Sigma(\Gamma_{p,2}(M)^{\frac{1}{p}})\} = \{\Gamma_{p,1}(M')^{\frac{1}{p}}, \Gamma_{p,2}(M')^{\frac{1}{p}}\}.$$

Hence  $\Gamma_{p,1}(M)^{\frac{1}{p}} \in \text{spn}(\Gamma_{p,1}(M')^{\frac{1}{p}}) \cup \text{spn}(\Gamma_{p,2}(M')^{\frac{1}{p}})$ . Therefore by Lemma 4.2.1,  $[M']$  and  $[M]$  are connected by a path in  $\mathfrak{S}_{L,p}(1)$ . Furthermore, as edges of the graph  $\mathfrak{S}_{L,p}(0)$ ,  $[M]$  and  $[M']$  are contained in the same connected component. Since the number of connected components in  $\mathfrak{S}_{L,p}(0)$  equals to  $g^+(\mathcal{G}_{L,p}(1))$  by Theorem 4.2.8, each spinor genus in  $\mathcal{G}_{L,p}(1)$  forms a connected

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component in  $\mathfrak{G}_{L,p}(1)$ . Furthermore, since  $g^+(\mathcal{G}_{L,p}(2m+1)) = g^+(\mathcal{G}_{L,p}(1))$ ,  $\text{spn}(\lambda_p^{\frac{m}{2}}(M)) = \text{spn}(\lambda_p^{\frac{m}{2}}(M'))$  if and only if  $\text{spn}(M) = \text{spn}(M')$  for any  $M, M' \in \mathcal{G}_{L,p}(2m+1)$ . Therefore by Lemma 4.2.4, the set of all vertices in the connected component of  $\mathfrak{G}_{L,p}(m)$  containing  $[M]$  is the set of equivalence classes in  $\text{Cspn}(M)$  for any odd  $m$ . The proof of even case is quite similar to this.  $\square$

**Lemma 4.2.10.** *Let  $[N] \in \mathcal{G}_{L,p}(m+1)$  be an edge of the graph  $\mathfrak{G}_{L,p}(m)$ . Then the set of all edges in the connected component of  $\mathfrak{G}_{L,p}(m)$  containing  $[N]$  is the set of all classes in  $\text{Cspn}(N)$ .*

*Proof.* It suffices to show that the set of edges in the connected component of  $\mathfrak{G}_{L,p}(m)$  containing  $[N]$  is exactly the set of vertices in the connected component of  $\mathfrak{G}_{L,p}(m+1)$  containing the vertex  $[N]$  by Lemma 4.2.9. Note that if  $N_1$  and  $N_2$  are different  $\Gamma_p$ -descendant of  $K$  for some  $K \in \mathcal{G}_{L,p}(m+2)$ , then  $\lambda_p(K)$  is a  $\Gamma_p$ -descendant of both  $N_1$  and  $N_2$ . This implies that every class in  $\text{Cspn}(N)$  is contained in the set of edges in the connected component of  $\mathfrak{G}_{L,p}(m)$  containing  $[N]$ . Conversely, assume that  $[N']$  is contained in the set of edges in the connected component of  $\mathfrak{G}_{L,p}(m)$  containing  $[N]$ . Without loss of generality, we may assume that there is a ternary  $\mathbb{Z}$ -lattice  $M$  that is a  $\Gamma_p$ -descendant of both  $N$  and  $N'$ . If  $m = 0$  or  $m \geq 1$  and  $\lambda_p(N) \neq \lambda_p(N')$ , then there is a  $\mathbb{Z}$ -lattice  $K$  whose  $\Gamma_p$ -descendants are both  $N$  and  $N'$  by Lemmas 4.2.1 and 4.2.2, that is, as vertices,  $[N]$  and  $[N']$  are contained in the edge  $[K]$ . Now suppose that  $\lambda_p(N) = \lambda_p(N')$ . Then in this case, there exists a ternary  $\mathbb{Z}$ -lattice  $S \in \mathcal{G}_{L,p}(m+1)$  such that  $\lambda_p(N) \neq \lambda_p(S)$  and  $M$  is a  $\Gamma_p$ -descendant of  $S$ . Hence there are edges containing  $\{[N], [S]\}$  and  $\{[S], [N']\}$ . This completes the proof.  $\square$

**Theorem 4.2.11.** *For any non-negative integer  $m$ , the graph  $\mathfrak{G}_{L,p}(m)$  has an odd cycle (including a loop) if and only if  $\mathfrak{G}_{L,p}(m)$  is of  $O$ -type.*

*Proof.* We already proved the case when  $m = 0$  in Theorem 4.2.8. Assume that  $m = 1$ . Let  $T \in \mathcal{G}_{L,p}(0)$  be any  $\mathbb{Z}$ -lattice. Then there are at least three  $\mathbb{Z}$ -lattices, say  $S_1, S_2, S_3$ , in  $\Phi_p(T) \cap \mathcal{G}_{L,p}(1)$ . Now by Lemma 4.2.1,  $[S_i]$  and  $[S_j]$  are connected by an edge for any  $1 \leq i \neq j \leq 3$ . Hence the graph  $\mathfrak{G}_{L,p}(1)$  contains a cycle of length 3 or a loop. For the general case, we may apply Lemma 4.2.4 to prove the theorem.  $\square$

### 4.3 Simplicity and Regularity

In this section, we assume that  $p$  is a prime and  $L$  is a positive definite non-classic ternary  $\mathbb{Z}$ -lattice on  $V$  satisfying the equation (4.1.1). We investigate some properties of the graph  $\mathfrak{G}_{L,p}(0)$ . As noted earlier, the graph  $\mathfrak{G}_{L,p}(0)$  might contain a loop. Let  $T$  be a  $\mathbb{Z}$ -lattice in  $\mathcal{G}_{L,p}(0)$ . One may easily verify that it has a loop if and only the trace of the Anzahlmatrix  $\pi_p(T)$  is not zero. Note that  $\pi_p(T)$  is independent of choices of the lattice  $T \in \mathcal{G}_{L,p}(0)$ . If the graph  $\mathfrak{G}_{L,p}(0)$  is of  $E$ -type, then it does not have any loop.

**Proposition 4.3.1.** *Let  $\alpha = 1$  (1 or 2) if  $p = 2$  ( $p > 2$ , respectively). Assume  $T \in \mathcal{G}_{L,p}(0)$ . There is a lattice  $\tilde{T} \in \mathcal{G}_{L,p}(0)$  such that  $o(\tilde{T}) < r(p\tilde{T}, \tilde{T})$  if the following conditions hold:*

- (a)  $\alpha np$  is represented by  $\mathcal{G}_{L,p}(0)$ ,
- (b)  $\langle \alpha np \rangle$  splits  $T_q$  for any prime  $q$  dividing  $n$ ,

where  $n$  is an integer not divisible by  $p$ . In particular, if  $dT$  or  $4 \cdot dT$  is squarefree, then  $\mathfrak{G}_{L,p}(0)$  has a loop.

*Proof.* By the first condition, there is a vector  $x \in V$  such that  $Q(x) = \alpha np$ . Note that there is an isometry  $\phi_q \in O(v_q)$  such that  $\mathbb{Z}_q \phi_q(x)$  splits  $T_q$  for any prime  $q$  dividing  $n$ . If we define  $U = \{q \mid \mathbb{Z}_q x \subseteq T_q, q \nmid n\}$ , then almost all primes are contained in  $U$ . For each prime  $q \notin U$  not dividing  $n$ , we fix an isometry  $\phi_q \in O(V_q)$  such that  $\phi_q(x) \in T_q$ . Now define a  $\mathbb{Z}$ -lattice  $\tilde{T} \in \mathcal{G}_{L,p}(0)$  such that

$$\tilde{T}_q = \begin{cases} T_q & \text{if } q \in U, \\ \phi_q^{-1}(T_q) & \text{otherwise.} \end{cases}$$

Then the vector  $x \in \tilde{T}$  satisfies  $B(x, \tilde{T}) \equiv 0 \pmod{n}$ . For any  $z \in \tilde{T}$ ,

$$\tau_x(pz) = pz - \frac{2B(x, pz)}{\alpha np} x \in \tilde{T}.$$

Therefore  $\tau_x(p\tilde{T}) \subseteq \tilde{T}$ . Since  $2B(x, \tilde{T}) \not\subseteq p\mathbb{Z}$ , we have  $\tau_x \notin O(\tilde{T})$ . Note that

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if  $dT$  or  $4 \cdot dT$  is squarefree, then  $p$  or  $2p$  is represented by the genus of  $T$ . Therefore one may easily show that there is a loop in  $\mathfrak{G}_{L,p}(0)$ .  $\square$

Recall that a graph is called *simple* if it has neither loops nor multiple edges between two vertices. A simple graph is called *k-regular* if every vertex has exactly  $k$  adjacent vertices. In the remaining of this section, we find an equivalent condition for  $\mathfrak{G}_{L,p}(0)$  to be a  $(p+1)$ -regular simple graph, when  $p$  is an odd prime. From now on we assume that  $p$  is an odd prime. Recall that the ordered set of all equivalence classes of  $\mathcal{G}_{L,p}(0)$  is

$$\{[T_1], [T_2], \dots, [T_h]\}$$

and the ordered set of all equivalence classes of  $\mathcal{G}_{L,p}(1)$  is

$$\{[S_1], [S_2], \dots, [S_k]\}.$$

Note that the graph  $\mathfrak{G}_{L,p}(0)$  does not have a loop if and only if

$$R(pT_i, T_i) = O(T_i)$$

for any  $T_i \in \mathcal{G}_{L,p}(0)$ .

Let  $U$  be a  $\mathbb{Z}$ -lattice such that

$$U_p \simeq \begin{pmatrix} 0 & \frac{p}{2} \\ \frac{p}{2} & 0 \end{pmatrix} \perp \langle \epsilon \rangle, \quad U_q \simeq L_q \quad (p \neq q).$$

Then one may easily show that the  $\lambda_p$ -transformation induces a bijection from the set of all equivalence classes of  $\mathcal{G}_{L,p}(1)$  to the set of all equivalence classes of  $\text{gen}(U)$ . We assume that the ordered set of equivalence classes of  $\text{gen}(U)$  is

$$\{[U_1], [U_2], \dots, [U_k]\},$$

where  $U_i = \lambda_p(S_i)$  for  $i = 1, 2, \dots, k$ . If a vertex  $[T_i]$  is adjacent to a vertex  $[T_j]$  by the edge  $[S_l]$ , then there are isometries  $\sigma, \tau \in O(V)$  such that  $S_l^{\frac{1}{p}} = \mathbb{Z}x_1 + \mathbb{Z}p^{-1}x_2 + \mathbb{Z}x_3$  and

$$\sigma(T_i) = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3, \quad \tau(T_j) = \mathbb{Z}px_1 + \mathbb{Z}p^{-1}x_2 + \mathbb{Z}x_3.$$



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Then  $\sigma(T_i) \cap \tau(T_j) = \mathbb{Z}px_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3 = \lambda_p(S) \in \text{gen}(U)$ . Conversely, if  $U' \in \text{gen}(U)$  is contained in  $\sigma(T_i)$  and  $\tau(T_j)$ , then  $\sigma(T_i) \cap \tau(T_j) = U'$ , for

$$[\sigma(T_i) : \sigma(T_i) \cap \tau(T_j)] = [\sigma(T_i) : U'] = p.$$

**Lemma 4.3.2.** *Suppose that  $p$  is an odd prime and  $R(pT_i, T_i) = O(T_i)$  for any  $1 \leq i \leq h$ . Then the graph  $\mathfrak{G}_{L,p}(0)$  is simple if and only if*

$$\frac{r(pT_i, T_j)}{o(T_i)} = \frac{o(T_j)}{o(\sigma(T_i) \cap T_j)}, \quad (4.3.1)$$

for any  $1 \leq i, j \leq h$  with  $r(pT_i, T_j) \neq 0$ , where the isometry  $\sigma \in O(V)$  satisfies  $\sigma(pT_i) \subset T_j$ .

*Proof.* We define a set

$$\Delta_p(T_j) = \{T' \in \mathcal{G}_{L,p}(0) - \{T_j\} \mid pT' \subset T_j\}.$$

Note that this set may contain a lattice that is isometric to  $T_j$ . We know that there are  $p+1$  lattices in  $\Delta_p(T_j)$ . Let  $T' \in \Delta_p(T_j)$ . Without loss of generality, we may assume that  $T' = T_i$  for some  $i = 1, 2, \dots, h$ . Furthermore since the equality holds for  $i = j$ , we may assume that  $i \neq j$ . Note that the number of lattices in  $\Delta_p(T_j)$  isometric to  $T_i$  is

$$\pi_p(T)_{ij} = \frac{r(pT_i, T_j)}{o(T_i)} \text{ (say } l\text{)}.$$

In fact, for the group action  $\Phi : R(pT_i, T_j) \times O(T_i) \mapsto R(pT_i, T_j)$  defined by  $\Phi(\sigma, \eta) = \sigma \circ \eta$ , the number of lattices in  $\Delta_p(T_j)$  isometric to  $T_i$  equals to the number of orbits in  $R(pT_i, T_j)$ . We assume that

$$\Delta_p(T_j) = \{\sigma_u(T_i) \mid \sigma_u \in R(pT_i, T_j)/O(T_i), 1 \leq u \leq l\}.$$

Note that for any  $\eta \in O(T_j)$ , since

$$p\eta(T_i) \subset \eta(pT_i) \subset \eta(T_j) = T_j,$$

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we have  $\eta(T_i) \in \Delta_p(T_j)$ . For  $\eta, \eta' \in O(T_j)$ , one may easily show that

$$\eta(T_i) = \eta'(T_i) \iff \eta^{-1}\eta' \in O(T_i \cap T_j).$$

Hence we have

$$\frac{o(T_j)}{o(T_i \cap T_j)} \leq \frac{r(pT_i, T_j)}{o(T_i)}.$$

First, suppose that  $\mathfrak{G}_{L,p}(0)$  is simple. Then there is an  $\eta_u \in O(V)$  such that

$$\eta_u(T_i \cap T_j) = \eta_u(T_i) \cap \eta_u(T_j) = \sigma_u(T_i) \cap T_j,$$

for each  $1 \leq u \leq l$ . Therefore we have  $\eta_u(T_i) = \sigma_u(T_i)$  and  $\eta_u \in O(T_j)$ .

Conversely, suppose that the equality (4.3.1) holds. Then we know that for any  $\sigma_u(T_i) \in \Delta_p(T_j)$ , there is an  $\eta \in O(T_j)$  such that  $\eta(T_i) = \sigma_u(T_i)$ . Hence  $\eta(T_i \cap T_j) = \sigma_u(T_i) \cap T_j$ , which implies that the number of edges between  $[T_i]$  and  $[T_j]$  is one.  $\square$

**Lemma 4.3.3.** *Let  $p$  be an odd prime. The graph  $\mathfrak{G}_{L,p}(0)$  is a  $(p+1)$ -regular simple graph if and only if  $R(p^2T_i, T_i) = O(T_i)$  for any  $i = 1, 2, \dots, h$ .*

*Proof.* Under the same notations and assumptions given above, if  $\sigma$  is an element of  $R(pT_i, T_j)$ , then

$$\sigma(p^2T_i) = p\sigma(pT_i) \subset pT_j \subset T_i.$$

Hence if  $R(p^2T_i, T_i) = O(T_i)$  for any  $i = 1, 2, \dots, h$ , then we know that  $\sigma \in R(p^2T_i, T_i) = O(T_i)$ . This implies that any  $p+1$  lattices in  $\Delta_p(T_j)$  are non isometric with each other for any  $1 \leq j \leq h$ . Therefore,  $\mathfrak{G}_{L,p}(0)$  is a  $(p+1)$ -regular simple graph.

Conversely, suppose that  $\mathfrak{G}_{L,p}(0)$  is  $(p+1)$ -regular and simple. Assume  $\sigma \in R(p^2T_j, T_j)$ . Since  $d(\sigma(p^2T_j)) = p^6d(T_j)$  and  $p \nmid 4 \cdot d(T_j)$ ,  $T_j/\sigma(p^2T_j)$  has three possibilities. If  $T_j/\sigma(p^2T_j)$  is isomorphic to

$$\mathbb{Z}/p^2\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z},$$

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then clearly  $\sigma(T_j) = T_j$ . If  $T_j/\sigma(p^2T_j)$  is isomorphic to

$$\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z},$$

then there is a basis  $\{x, y, z\}$  for  $T_j$  such that

$$T_j = \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}z \quad \text{and} \quad \sigma(p^2T_j) = \mathbb{Z}px + \mathbb{Z}p^2y + \mathbb{Z}p^3z.$$

Since there is no loops in  $\mathfrak{G}_{L,p}(0)$ ,  $\sigma \in O(T_j)$ . Finally, if  $T_j/\sigma(p^2T_j)$  is isomorphic to

$$\mathbb{Z}/p^2\mathbb{Z} \oplus \mathbb{Z}/p^4\mathbb{Z},$$

then there is a basis  $\{x, y, z\}$  for  $T_j$  such that

$$T_j = \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}z \quad \text{and} \quad \sigma(p^2T_j) = \mathbb{Z}x + \mathbb{Z}p^2y + \mathbb{Z}p^4z.$$

If we define  $\tilde{T} = \mathbb{Z}p^{-1}x + \mathbb{Z}y + \mathbb{Z}pz \in \text{gen}(T)$ , then  $T_j, \sigma(T_j) \in \Delta_p(\tilde{T})$ . Since every lattice in  $\Delta_p(\tilde{T})$  is non-isometric to each other by assumption,  $\sigma(T_j) = T_j$ . The lemma follows from this.  $\square$

**Lemma 4.3.4.** *Let  $p$  be an odd prime. Suppose that  $\mathfrak{G}_{L,p}(0)$  is a  $(p+1)$ -regular simple graph. Then  $o(T_i) = o(S_j) = 2$ , for any  $1 \leq i \leq h$  and  $1 \leq j \leq k$ .*

*Proof.* Assume that two end points of the edge  $[S_l]$  are  $[T_i]$  and  $[T_j]$ . Then, without loss of generality, we may assume that there is a basis  $\{x, y, z\}$  for  $T_j$  such that

$$T_j = \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}z \quad \text{and} \quad T_j = \mathbb{Z}px + \mathbb{Z}p^{-1}y + \mathbb{Z}z.$$

For any  $\sigma \in O(T_j)$ , we know that  $p\sigma(T_i) = \sigma(pT_i) \subset \sigma(T_j) = T_j$ , that is,  $\sigma(T_i) \in \Delta_p(T_j)$ . Since  $\mathfrak{G}_p(T)$  is  $(p+1)$ -regular simple,  $\sigma(T_i) = T_i$ . Therefore we have  $O(T_i) = O(T_j)$ .

Suppose that  $O(T_j) \neq 2$ . Since  $O(T_j)$  is generated by symmetries and  $-I$  (see [5]), there is a symmetry  $\tau_{x_1} \in O(T_j)$ , where  $x_1$  is a primitive vector of  $T_j$ . Then one may easily show that there is a basis for  $T_j$  such that

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$T_j = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3$  with

$$B(x_i, x_j) = \begin{pmatrix} Q(x_1) & a & b \\ a & p^2c & pd \\ b & pd & e \end{pmatrix},$$

where  $a, b, c, d$  and  $e$  are contained in  $\frac{1}{2}\mathbb{Z}$ . If we define

$$\tilde{T} = \mathbb{Z}px_1 + \mathbb{Z}p^{-1}x_2 + \mathbb{Z}x_3 \in \text{gen}(T),$$

then  $\tilde{T} \in \Delta_p(T_j)$ . By the above observation, we know that  $\tau_{x_1} = \tau_{px_1} \in O(\tilde{T})$ . This is a contradiction to the fact that  $4 \cdot d(T_j)$  is not divisible by  $p$ . Therefore the isometry group of any lattice in the genus of  $T$  is trivial. Now by Lemma 4.3.2,  $O(T_i \cap T_j)$  is also trivial. As mentioned earlier,  $\lambda_p(S_l) \simeq T_i \cap T_j$ . This completes the proof.  $\square$

Combining all these lemmas, we have the following theorem.

**Theorem 4.3.5.** *Assume that for an odd prime  $p$ , a ternary  $\mathbb{Z}$ -lattice  $L$  satisfies the equation (4.1.1). The graph  $\mathfrak{G}_{L,p}(0)$  is a  $(p+1)$ -regular simple graph if and only if  $r(p^2T_i, T_i) = o(T_i) = 2$  for any  $1 \leq i \leq h$ .*

**Remark 4.3.6.** Assume that for an odd prime  $p$ , a ternary  $\mathbb{Z}$ -lattice  $L$  satisfies the equation (4.1.1).

- (a) If  $\mathfrak{G}_{L,p}(0)$  is  $(p+1)$ -regular simple, then the Eichler's Anzahlmatrix  $\pi_p(T)$  is the adjacent matrix of  $\mathfrak{G}_{L,p}(0)$ .
- (b) If  $\mathfrak{G}_{T,p}(0)$  is  $(p+1)$ -regular simple, then for any  $\mathbb{Z}$ -lattice  $N$  such that  $\lambda_q(N) \simeq T$  for a prime  $q \neq p$ , the graph  $\mathfrak{G}_{N,p}(0)$  is also  $(p+1)$ -regular simple. This follows from the fact that the map

$$R(p^n N, N) \mapsto R(p^n \Lambda_q(N), \Lambda_q(N))$$

is injective (see [5]).

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(c) Let  $T$  be a ternary  $\mathbb{Z}$ -lattice whose matrix presentation is

$$\begin{pmatrix} 4 & 1 & 1 \\ 1 & 14 & 3 \\ 1 & 3 & 388 \end{pmatrix}.$$

Then one may easily show that  $h(T) = 15$  and  $R(3^2T', T') = O(T') = 2$  for any  $T' \in \text{gen}(T)$ . Hence  $\mathfrak{G}_{T,3}(0)$  is a regular quartic graph by the above theorem.

## Chapter 5

# Arithmetic relations of the representations of ternary quadratic forms

Throughout this chapter, we assume that a ternary  $\mathbb{Z}$ -lattice  $L$  and a prime  $p$  satisfy all conditions given in chapter 4. For a nonnegative integer  $m$ , let  $T \in \mathcal{G}_{L,p}(m)$  be a ternary  $\mathbb{Z}$ -lattice and let  $S \in \mathcal{G}_{L,p}(m+1)$  be a ternary  $\mathbb{Z}$ -lattice such that  $r(T^p, S) \neq 0$ . This implies that  $[T]$  is one of vertices contained in the edge  $[S]$  in the graph  $\mathfrak{G}_{L,p}(m)$ . We assume that

$$\text{Cspn}(T) = \{[T_1], [T_2], \dots, [T_u]\} \text{ and } \text{Cspn}(S) = \{[S_1], [S_2], \dots, [S_v]\} \quad (5.0.1)$$

are *ordered* sets of equivalence classes. The aim of this chapter is to show that if  $m \leq 2$ , then there are rational numbers  $a_i$  and  $b_i$  such that for any integer  $n$  (any integer  $n$  divisible by  $p$  only when  $m = 2$ ),

$$r(n, T) = \sum_{i=1}^v (a_i r(pn, S_i) + b_i r(p^3 n, S_i)) + (\text{some extra term}). \quad (5.0.2)$$

This chapter is a part of [13], we bring it here intactly.

## 5.1 The case when $m = 0$

In this section, we prove the equation 5.0.2 in case of  $m = 0$ . In some cases, the extra term in the above equation can be removed. The types of the graph  $\mathbb{G}_{L,p}(0)$  defined in chapter 4 is used to determine when it happens.

For a while, we assume that  $m$  is an arbitrary nonnegative integer. The following two propositions will be used repeatedly.

**Proposition 5.1.1.** *For any integer  $n$ ,*

$$\frac{r(pn, S)}{o(S)} = \sum_{i=1}^u \frac{r(T_i^p, S)}{o(S)} \frac{r(n, T_i)}{o(T_i)} - \frac{r(pn, \Lambda_p(S))}{o(S)}.$$

*Proof.* By Weak Approximation Theorem for Rotations, there exists a basis  $\{x_1, x_2, x_3\}$  for  $S$  such that

$$(B(x_i, x_j)) \equiv \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \perp \langle p^{m+1}\delta \rangle \pmod{p^{m+2}},$$

where  $\delta$  is an integer not divisible by  $p$ . As in Lemma 4.1.1, let

$$\Gamma_{p,1}(S) = \mathbb{Z}px_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3, \quad \Gamma_{p,2}(S) = \mathbb{Z}x_1 + \mathbb{Z}px_2 + \mathbb{Z}x_3.$$

Since  $Q(x) \equiv a_1a_2 \pmod{p}$  for any  $x = a_1x_1 + a_2x_2 + a_3x_3 \in S$ , we have  $Q(x) \equiv 0 \pmod{p}$  if and only if  $a_1 \equiv 0 \pmod{p}$  or  $a_2 \equiv 0 \pmod{p}$ . Hence

$$x \in R(pn, S) \quad \text{if and only if} \quad x \in R(pn, \Gamma_{p,1}(S)) \cup R(pn, \Gamma_{p,2}(S))$$

Furthermore since  $\Gamma_{p,1}(S) \cap \Gamma_{p,2}(S) = \Lambda_p(S)$ , we have

$$r(pn, S) = r(pn, \Gamma_{p,1}(S)) + r(pn, \Gamma_{p,2}(S)) - r(pn, \Lambda_p(S))$$

for any integer  $n$ . Note that  $\Gamma_{p,1}(S)$  and  $\Gamma_{p,2}(S) \in \text{gen}(T^p)$  are the only sublattices of  $S$  that are contained in  $\text{gen}(T^p)$ . Furthermore, since the edge  $[S]$  in  $\mathfrak{G}_{L,p}(0)$  contains the vertex  $[T]$  by assumption, we have

$$\Gamma_{p,1}(S)^{\frac{1}{p}}, \Gamma_{p,2}(S)^{\frac{1}{p}} \in \text{Cspn}(T).$$

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Now for any  $\mathbb{Z}$ -lattice  $T_i \in \text{Cspn}(T)$ , the number of sublattices in  $S$  that are isometric to  $T_i^p$  is  $\frac{r(T_i^p, S)}{o(T_i)}$ . The proposition follows from this.  $\square$

**Proposition 5.1.2.** *For any integer  $n$ ,*

$$\frac{r(pn, T)}{o(T)} = \begin{cases} \sum_{j=1}^v \frac{r(S_j^p, T)}{o(T)} \frac{r(n, S_j)}{o(S_j)} - p \cdot \frac{r(n, T^p)}{o(T)} & \text{if } m = 0, \\ \sum_{j=1}^v \frac{\tilde{r}(S_j^p, T)}{o(T)} \frac{r(n, S_j)}{o(S_j)} + \frac{r(pn, \Lambda_p(T))}{o(T)} - 2p \cdot \frac{r(n, T^p)}{o(T)} & \text{otherwise.} \end{cases}$$

*Proof.* If we take  $\epsilon = 0$  and  $L = T$  in Lemma 3.2.5, then we have

$$r(pn, T) = \sum_{M \in \Omega_p(0, T)} r(pn, M) - (s_p(0, T) - 1)r(n, T^p).$$

First, assume that  $m = 0$ . Let  $M \in \Omega_p(0, T)$  be a  $\mathbb{Z}$ -lattice. Then by Lemmas 3.2.3 and 3.2.4,

$$M_p \simeq \begin{pmatrix} 0 & \frac{p}{2} \\ \frac{p}{2} & 0 \end{pmatrix} \perp \langle -4p^2 dT \rangle \quad \text{and} \quad M_q \simeq T_q \quad (q \neq p).$$

Hence  $M \in \text{gen}(S^p)$ . Furthermore, since  $r(T^p, M^{\frac{1}{p}}) = \tilde{r}(M, T) \neq 0$  and  $r(T^p, S) = \tilde{r}(S^p, T) \neq 0$  by Lemma 3.2.7,  $M^{\frac{1}{p}} \in \text{Cspn}(S)$  by Lemmas 4.2.1 and 4.2.9. Conversely, if  $M^{\frac{1}{p}} \in \text{Cspn}(S)$  satisfies  $\tilde{r}(M, T) \neq 0$ , then  $M$  is isometric to a  $\mathbb{Z}$ -lattice in  $\Omega_p(0, T)$ . Note that the number of lattices in  $\Omega_p(0, T)$  that are isometric to  $S^p$  is  $\frac{r(S^p, T)}{o(S)}$  and  $s_p(0, T) = p + 1$ . The proof of the case when  $m \geq 1$  is quite similar to this, except that there is a unique  $\mathbb{Z}$ -lattice in  $\Omega_p(0, T)$  that is not contained in  $\text{gen}(S^p)$ , which is, in fact,  $\Lambda_p(T)$ , and  $s_p(0, T) = 2p + 1$ .  $\square$

We define

$$\mathcal{M}_{L,p}(m) = \left( \frac{r(T_i^p, S_j)}{o(T_i)} \right) \in M_{u,v}(\mathbb{Z}) \quad \text{and} \quad \mathcal{N}_{L,p}(m) = \left( \frac{r(T_i^p, S_j)}{o(S_j)} \right) \in M_{u,v}(\mathbb{Z}).$$

Note that these two matrices depend on the order of each set  $\text{Cspn}(\cdot)$ , and  $\mathcal{M}_{L,p}(0)$  is one of block diagonal components of  $\mathfrak{M}_{L,p}(0)$  if we take a suitable



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order in (4.2.1). For any integer  $n$ , we define vectors

$$\mathbf{R}(n, \text{Cspn}(T)) = \left( \frac{r(n, T_1)}{o(T_1)}, \frac{r(n, T_2)}{o(T_2)}, \dots, \frac{r(n, T_u)}{o(T_u)} \right)^t,$$

$$\mathbf{R}^\sharp(n, \text{Cspn}(\lambda_p^m(T))) = \left( \frac{r(n, \lambda_p^m(T_1))}{o(T_1)}, \frac{r(n, \lambda_p^m(T_2))}{o(T_2)}, \dots, \frac{r(n, \lambda_p^m(T_u))}{o(T_u)} \right)^t.$$

Similarly, we define  $\mathbf{R}(n, \text{Cspn}(S))$  and  $\mathbf{R}^\sharp(n, \text{Cspn}(\lambda_p^m(S)))$ . If  $\text{Cspn}(M)$  is  $\text{spn}(M)$ , then we use  $\mathbf{R}(n, \text{spn}(M))$  rather than  $\mathbf{R}(n, \text{Cspn}(M))$ .

**Theorem 5.1.3.** *Let  $T$  and  $S$  be ternary  $\mathbb{Z}$ -lattices satisfying all conditions given above when  $m = 0$ . If the graph  $\mathfrak{G}_{L,p}(0)$  is of  $O$ -type, then we have*

$$p\mathbf{R}(n, \text{spn}(T^p)) = \mathcal{M} \cdot \mathbf{R}(n, \text{spn}(S)) - \mathcal{Z} \cdot (\mathbf{R}(p^2n, \text{spn}(S)) + \mathbf{R}(n, \text{spn}(S))),$$

where  $\mathcal{Z} = (\mathcal{M} \cdot \mathcal{N}^t)^{-1} \mathcal{M}$ .

*Proof.* By Lemma 4.2.6 and Propositions 5.1.1, 5.1.2, we have the following two equalities:

$$\mathbf{R}(pn, \text{spn}(S)) = \mathcal{N}^t \cdot \mathbf{R}(n, \text{spn}(T)) - \mathbf{R}^\sharp(pn, \text{spn}(\Lambda_p(S))), \quad (5.1.1)$$

$$\mathbf{R}(pn, \text{spn}(T)) = \mathcal{M} \cdot \mathbf{R}(n, \text{spn}(S)) - p\mathbf{R}(n, \text{spn}(T^p)). \quad (5.1.2)$$

Since  $\lambda_p(\lambda_p(S_i)) \simeq S_i$  for any  $S_i \in \text{spn}(S)$ , we have

$$\mathbf{R}^\sharp(p^2n, \text{spn}(\Lambda_p(S))) = \mathbf{R}(n, \text{spn}(S)).$$

Hence

$$\mathbf{R}(p^2n, \text{spn}(S)) = \mathcal{N}^t \cdot \mathbf{R}(pn, \text{spn}(T)) - \mathbf{R}(n, \text{spn}(S)). \quad (5.1.3)$$

Note that

$$\mathbf{O}(\text{spn}(T)) \cdot \mathcal{N} = \mathcal{M} \cdot \mathbf{O}(\text{spn}(S)),$$

where  $\mathbf{O}(\text{spn}(T))$  is the  $u \times u$  diagonal matrix with entries  $o(T_i)^{-1}$ . Furthermore, since we are assuming that  $\text{rank}(\mathcal{M}) = u$ , the  $u \times u$  square matrix  $\mathcal{M} \cdot \mathcal{N}^t$  is invertible. Therefore the equation follows directly from (5.1.2) and

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(5.1.3). □

Now assume that  $\mathfrak{G}_{L,p}(0)$  is of  $E$ -type, then  $\text{Cspn}(T)$  consists of two spinor genera and each connected component is a bipartite graph. Hence the rank of the matrix  $\mathcal{M}$  is  $u - 1$  and  $\mathcal{M} \cdot \mathcal{N}^t$  is no longer invertible. To get a similar result for an  $E$ -type graph, we need to make some adjustments.

Assume that  $\text{Cspn}(T) = \text{spn}(T) \cup \text{spn}(\tilde{T})$  and

$$\text{spn}(T) = \{[T_{i_1}], \dots, [T_{i_a}]\}, \quad \text{spn}(\tilde{T}) = \{[T_{j_1}], \dots, [T_{j_b}]\},$$

where  $\{i_1, i_2, \dots, i_a, j_1, \dots, j_b\} = \{1, 2, \dots, u\}$ . Note that

$$w(\text{spn}(T')) = \sum_{[K] \in \text{spn}(T')} \frac{1}{o(K)},$$

is independent of  $T'$  for any  $T' \in \text{gen}(T)$ . Define

$$\epsilon_l = \begin{cases} w(\text{spn}(T))^{-1} & \text{if } l \in \{i_1, \dots, i_a\}, \\ -w(\text{spn}(T))^{-1} & \text{if } l \in \{j_1, \dots, j_b\}, \end{cases}$$

and define a  $u \times (v + 1)$  matrix  $\tilde{\mathcal{N}} = (n_{ij})$  by

$$n_{ij} = \begin{cases} \frac{r(T_i^p, S_j)}{o(S_j)} & \text{if } j \leq v, \\ \epsilon_i & \text{if } j = v + 1. \end{cases}$$

**Lemma 5.1.4.** *The rank of the matrix  $\tilde{\mathcal{N}}$  defined above is  $u$ .*

*Proof.* Let  $\mathbf{n}_i$  be the  $i$ -th row vector of the matrix  $\tilde{\mathcal{N}}$ . Suppose that

$$\alpha_1 \mathbf{n}_1 + \dots + \alpha_u \mathbf{n}_u = 0$$

for some integers  $\alpha_i$ , that is,

$$\begin{cases} \alpha_1 \frac{r(T_1^p, S_j)}{o(S_j)} + \dots + \alpha_u \frac{r(T_u^p, S_j)}{o(S_j)} = 0 & \text{for any } j = 1, \dots, v, \\ \alpha_1 \epsilon_1 + \dots + \alpha_u \epsilon_u = 0. \end{cases} \quad (5.1.4)$$

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For any  $j$  such that  $1 \leq j \leq v$ , the edge named by  $[S_j]$  contains two vertices, one of them, say  $[T_{i_e}]$ , is contained in  $\text{spn}(T)$  and the other, say  $[T_{j_f}]$ , is contained in  $\text{spn}(\tilde{T})$ . Hence the first equation in (5.1.4) implies that

$$\alpha_{i_e} \frac{r(T_{i_e}^p, S_j)}{o(S_j)} + \alpha_{j_f} \frac{r(T_{j_f}^p, S_j)}{o(S_j)} = 0.$$

Therefore  $\alpha_{i_e} \cdot \alpha_{j_f} \leq 0$ . Since the subgraph of  $\mathfrak{G}_{L,p}(0)$  consisting of vertices in  $\text{Cspn}(T)$  is a connected bipartite graph, each  $\alpha_{i_e}$  ( $\alpha_{j_f}$ ) is 0, or it has the same sign to  $\alpha_{i_1}$  ( $\alpha_{j_1}$ , respectively). Therefore  $\alpha_l = 0$  for any  $l = 1, \dots, u$  and  $\text{rank}(\tilde{\mathcal{N}}) = u$ . This completes the proof.  $\square$

For a vector  $\mathbf{v} = (v_1, \dots, v_n)$ , we define

$$(\mathbf{v}, w_1, \dots, w_s) = (v_1, \dots, v_n, w_1, \dots, w_s).$$

Note that the equation (5.1.3) implies that

$$\tilde{\mathbf{R}} := \tilde{\mathcal{N}}^t \cdot \mathbf{R}(pn, \text{Cspn}(T)) = \begin{pmatrix} \mathbf{R}(p^2n, \text{spn}(S)) + \mathbf{R}(n, \text{spn}(S)) \\ r(pn, \text{spn}(T)) - r(pn, \text{spn}(\tilde{T})) \end{pmatrix}, \quad (5.1.5)$$

where

$$r(pn, \text{spn}(T)) = \frac{1}{w(\text{spn}(T))} \cdot \sum_{[T_i] \in \text{spn}(T)} \frac{r(pn, T_i)}{o(T_i)}.$$

**Theorem 5.1.5.** *If  $\mathfrak{G}_{L,p}(0)$  is of  $E$ -type, then we have*

$$p\mathbf{R}(n, \text{Cspn}(T^p)) = \mathcal{M} \cdot \mathbf{R}(n, \text{spn}(S)) - (\tilde{\mathcal{N}} \cdot \tilde{\mathcal{N}}^t)^{-1} \tilde{\mathcal{N}} \cdot \tilde{\mathbf{R}}.$$

*Proof.* From the above lemma, we know that  $\text{rank}(\tilde{\mathcal{N}}) = u$ . The theorem follows directly from the equations (5.1.2) and (5.1.5).  $\square$

Note that  $r(pn, \text{spn}(T)) - r(pn, \text{spn}(\tilde{T}))$  can easily be computed by the formula given in [20].

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**Example 5.1.6.** *Let  $p = 11$  and  $L = \langle 1, 1, 16 \rangle$ . Then we know that*

$$\mathcal{G}_{L,p}(0) = \left\{ T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{pmatrix}, T_2 = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 5 \end{pmatrix} \right\},$$

$$\mathcal{G}_{L,p}(1) = \left\{ S_1 = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 6 & -1 \\ 1 & -1 & 11 \end{pmatrix}, S_2 = \begin{pmatrix} 6 & 2 & 3 \\ 2 & 6 & 1 \\ 3 & 1 & 7 \end{pmatrix} \right\},$$

*up to isometry. One may easily compute that  $\mathcal{M} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$  and  $\mathcal{N} = \begin{pmatrix} 8 & 4 \\ 8 & 4 \end{pmatrix}$ . Since  $\text{rank}(\mathcal{M}) = 1$ , the graph  $\mathfrak{G}_{L,p}(0)$  is of  $E$ -type by Theorem 4.2.8. Note that  $\tilde{\mathcal{N}} = \begin{pmatrix} 8 & 4 & 16 \\ 8 & 4 & -16 \end{pmatrix}$ . Therefore, by Theorem 5.1.5, we have*

$$\begin{aligned} 11r(n, T_1^{11}) &= \frac{38}{5}r(n, S_1) - \frac{2}{5}r(11^2n, S_1) + \frac{39}{10}r(n, S_2) - \frac{1}{10}r(11^2n, S_2) \\ &\quad - \left( \frac{1}{2}r(11n, T_1) - \frac{1}{2}r(11n, T_2) \right), \\ 11r(n, T_2^{11}) &= \frac{38}{5}r(n, S_1) - \frac{2}{5}r(11^2n, S_1) + \frac{39}{10}r(n, S_2) - \frac{1}{10}r(11^2n, S_2) \\ &\quad + \left( \frac{1}{2}r(11n, T_1) - \frac{1}{2}r(11n, T_2) \right). \end{aligned}$$

*Note that by Korollar 2 of [20], one may easily check that*

$$r(11n, T_1) - r(11n, T_2) = \begin{cases} 0 & \text{if } n \neq 11m^2, \\ \left( \frac{1 - (-1)^m}{2} \right) \cdot (-1)^{\frac{m+1}{2}} \cdot 44m & \text{if } n = 11m^2. \end{cases}$$

## 5.2 The case when $m \geq 1$

In this section we prove the equation (5.0.2) in case of  $m = 1$ . For the case when  $m = 2$ , we give an example such that the above statement does not hold, and prove that the above statement still holds for  $m = 2$  if we additionally assume that  $n$  is divisible by  $p$ . In the case when  $m \geq 3$ , we

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show that under some restriction, the above statement holds if we replace  $r(n, T)$  by  $r(p^2 n, T) - pr(n, T)$ , and for any integer  $n$  not divisible by  $p$ , both  $r(n, T)$  and  $r(pn, T)$  can be written as a linear summation of  $r(pn, S)$ 's and  $r(n, S)$ 's, respectively, for  $S \in \mathcal{G}_{L,p}(m+1)$ .

**Theorem 5.2.1.** *Let  $T \in \mathcal{G}_{L,p}(1)$  and  $S \in \mathcal{G}_{L,p}(2)$  be ternary  $\mathbb{Z}$ -lattices satisfying  $r(T^p, S) \neq 0$ . Then we have*

$$(3p^2 - p) \cdot r(n, T) = \sum_{[\tilde{S}] \in \text{gen}(S)} \frac{\tilde{r}(\tilde{S}^p, T)}{o(\tilde{S})} \left( \frac{3p}{2} r(pn, \tilde{S}) - \frac{p}{p-1} r(p^3 n, \tilde{S}) \right) \\ + \frac{1}{p-1} \left( o(\Gamma_{p,1}(T)) \sum_{\substack{[\tilde{S}] \in \text{gen}(S) \\ \lambda_p(\tilde{S}) \simeq \Gamma_{p,1}(T)^{\frac{1}{p}}}} \frac{r(p^3 n, \tilde{S})}{o(\tilde{S})} + o(\Gamma_{p,2}(T)) \sum_{\substack{[\tilde{S}] \in \text{gen}(S) \\ \lambda_p(\tilde{S}) \simeq \Gamma_{p,2}(T)^{\frac{1}{p}}}} \frac{r(p^3 n, \tilde{S})}{o(\tilde{S})} \right).$$

*Proof.* First, we assume that

$$\Phi_p(\lambda_p(S)) = \{T = T_1, T_2, \dots, T_{p+1}\}$$

and

$$\Psi_p(\lambda_p(S)) = \{S = S_1, S_2, \dots, S_{\frac{p(p+1)}{2}}\}.$$

Without loss of generality, we may assume that  $\lambda_p(S) = \Gamma_{p,1}(T)^{\frac{1}{p}}$ . Define, for any integer  $n$ ,

$$\mathbf{R}(n, \Phi_p(\lambda_p(S))) = (r(n, T_1), r(n, T_2), \dots, r(n, T_{p+1}))^t$$

and

$$\mathbf{R}(n, \Psi_p(\lambda_p(S))) = \left( r(n, S_1), r(n, S_2), \dots, r\left(n, S_{\frac{p(p+1)}{2}}\right) \right)^t.$$

We also define a vector  $\mathbf{I}(n, \lambda_p(S)) = r(n, \lambda_p(S)) \cdot (1, 1, \dots, 1)^t$  of length  $\frac{p(p+1)}{2}$ . Now by Proposition 5.1.1, we have

$$\mathbf{R}(pn, \Psi_p(\lambda_p(S))) = U \cdot \mathbf{R}(n, \Phi_p(\lambda_p(S))) - \mathbf{I}\left(\frac{n}{p}, \lambda_p(S)\right),$$

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where  $U^t \in M_{(p+1) \times \frac{p(p+1)}{2}}(\mathbb{Z})$  is the incidence matrix of the complete graph of order  $p+1$  by Lemma 4.2.1. Therefore  $U^t U = (p-1)I + J$  and

$$((U^t U)^{-1} U^t)_{ij} = \begin{cases} \frac{1}{p} & \text{if } r(T_i^p, S_j) \neq 0, \\ \frac{-1}{p(p-1)} & \text{if } r(T_i^p, S_j) = 0. \end{cases}$$

Here  $J$  is a matrix of ones. Therefore we have

$$r(n, T) = \frac{1}{p} \sum_1 r(pn, S) - \frac{1}{p(p-1)} \sum_2 r(pn, S) + \frac{1}{2} r\left(\frac{n}{p}, \lambda_p(S)\right), \quad (5.2.1)$$

where  $\sum_1$  is the summation of all lattices  $S'$  contained in  $\Psi_p(\lambda_p(S))$  such that  $r(T^p, S') \neq 0$  and  $\sum_2$  is the summation of all lattices  $S'$  in  $\Psi_p(\lambda_p(S))$  such that  $r(T^p, S') = 0$ . We define, for simplicity,  $U_1(pn, S) = \sum_1 r(pn, S)$  and  $U_2(pn, S) = \sum_2 r(pn, S)$ . Now, by Proposition 3.2.9, we have

$$\begin{aligned} pr(pn, \lambda_p(S)) + \frac{p(p-1)}{2} r\left(\frac{n}{p}, \lambda_p(S)\right) &= o(\lambda_p(S)) r(pn, \text{gen}_p^{\lambda_p(S)}(S)) \\ &= \sum_{i=1}^{\frac{p(p+1)}{2}} r(pn, S_i) \\ &= U_1(pn, S) + U_2(pn, S). \end{aligned} \quad (5.2.2)$$

Let  $\tilde{S}$  be a  $\mathbb{Z}$ -lattice such that  $\lambda_p(\tilde{S}) = \Gamma_{p,2}(T)^{\frac{1}{p}}$ . We may similarly define  $\mathbf{R}(n, \Psi_p(\lambda_p(\tilde{S})))$ ,  $U_1(pn, \tilde{S})$  and  $U_2(pn, \tilde{S})$ . Then, equations (5.2.1) and (5.2.2) hold even if we replace  $S$  by  $\tilde{S}$ . Furthermore, by Proposition 5.1.2,

$$\begin{aligned} r(p^2 n, T) + (2p-1)r(n, T) &= \sum_{[S'] \in \text{gen}(S)} \frac{\tilde{r}((S')^p, T)}{o(S')} r(pn, S') \\ &= U_1(pn, S) + U_1(pn, \tilde{S}). \end{aligned} \quad (5.2.3)$$

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By combining (5.2.1)~(5.2.3), we have

$$\begin{aligned} \frac{3p^2-p}{2}r(n, T) &= p(U_1(pn, S) + U_1(pn, \tilde{S})) - p\left(\frac{1}{p}U_1(p^3n, S) - \frac{1}{p(p-1)}U_2(p^3n, S)\right) \\ &\quad - \frac{p(p-1)}{2}\left(\frac{1}{p}U_1(pn, S) - \frac{1}{p(p-1)}U_2(pn, S)\right) - \frac{1}{2}(U_1(pn, S) + U_2(pn, S)) \\ &= \frac{p}{2}U_1(pn, S) + pU_1(pn, \tilde{S}) - \left(U_1(p^3n, S) - \frac{1}{p-1}U_2(p^3n, S)\right). \end{aligned}$$

Since the above equation holds even if we exchange  $S$  for  $\tilde{S}$ , we have

$$\begin{aligned} (3p^2 - p)r(n, T) &= \frac{3p}{2}\left(U_1(pn, S) + U_1(pn, \tilde{S})\right) - \frac{p}{p-1}\left(U_1(p^3n, S) + U_1(p^3n, \tilde{S})\right) \\ &\quad + \frac{1}{p-1}\left(U_1(p^3n, S) + U_2(p^3n, S) + U_1(p^3n, \tilde{S}) + U_2(p^3n, \tilde{S})\right). \end{aligned}$$

This completes the proof.  $\square$

**Remark 5.2.2.** In the above theorem, one may easily check that the sets  $\Psi_p(\lambda_p(S))$  and  $\Psi_p(\lambda_p(\tilde{S}))$  are contained in  $\text{Cspn}(S)$ .

Assume that  $m = 2$ . Recall that  $T \in \mathcal{G}_{L,p}(2)$  and  $S \in \mathcal{G}_{L,p}(3)$  are ternary  $\mathbb{Z}$ -lattices satisfying  $r(T^p, S) \neq 0$ . If we define  $\epsilon_l$  and  $\tilde{\mathcal{N}}$  as before for the  $E$ -type, then Lemma 5.1.4 still holds under this situation.

**Theorem 5.2.3.** *Let  $T$  and  $S$  be ternary  $\mathbb{Z}$ -lattices satisfying all conditions given above. Assume that the graph  $\mathfrak{G}_{L,p}(2)$  is of  $O$ -type. If  $n$  is not divisible by  $p$ , then we have*

$$\mathbf{R}(n, \text{spn}(T)) = (\mathcal{N} \cdot \mathcal{N}^t)^{-1} \mathcal{N} \cdot \mathbf{R}(pn, \text{spn}(S)). \quad (5.2.4)$$

If  $n$  is divisible by  $p$ , then  $\mathbf{R}(n, \text{spn}(T))$  is equal to

$$\frac{1}{2p-1} \left( \mathcal{M} \cdot \mathbf{R}(pn, \text{spn}(S)) - (\mathcal{N} \cdot \mathcal{N}^t)^{-1} \mathcal{N} \cdot (\mathbf{R}(pn, \text{spn}(S)) + \mathbf{R}(p^3n, \text{spn}(S))) \right).$$

If  $\mathfrak{G}_{L,p}(2)$  is of  $E$ -type, then we have

$$\mathbf{R}(n, \text{Cspn}(T)) = \begin{cases} (\tilde{\mathcal{N}} \cdot \tilde{\mathcal{N}}^t)^{-1} \tilde{\mathcal{N}} \cdot \tilde{\mathbf{R}}_1 & \text{if } p \nmid n, \\ \frac{1}{2p-1} \left( \mathcal{M} \cdot \mathbf{R}(pn, \text{spn}(S)) - (\tilde{\mathcal{N}} \cdot \tilde{\mathcal{N}}^t)^{-1} \tilde{\mathcal{N}} \cdot \tilde{\mathbf{R}}_2 \right) & \text{otherwise,} \end{cases}$$

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where

$$\tilde{\mathbf{R}}_1 = \begin{pmatrix} \mathbf{R}(pn, \operatorname{spn}(S)) \\ r(n, \operatorname{spn}(T)) - r(n, \operatorname{spn}(\tilde{T})) \end{pmatrix},$$

and

$$\tilde{\mathbf{R}}_2 = \begin{pmatrix} \mathbf{R}(pn, \operatorname{spn}(S)) + \mathbf{R}(p^3n, \operatorname{spn}(S)) \\ (2p-1)(r(n, \operatorname{spn}(\tilde{T})) - r(n, \operatorname{spn}(T))) \end{pmatrix}.$$

*Proof.* The proof is similar to that of Theorem 5.1.3. First, assume that  $\mathfrak{G}_{L,p}(2)$  is of  $O$ -type. Since the rank of  $\mathcal{N}$  is  $u$ , we may define

$$\mathcal{Z} = (\mathcal{N} \cdot \mathcal{N}^t)^{-1} \mathcal{N}.$$

From the equation (5.1.1), we have

$$\mathbf{R}(n, \operatorname{spn}(T)) = \mathcal{Z} \left( \mathbf{R}(pn, \operatorname{spn}(S)) + \mathbf{R}^\sharp \left( \frac{n}{p}, \operatorname{spn}(\lambda_p(S)) \right) \right), \quad (5.2.5)$$

and

$$\mathbf{R}(p^2n, \operatorname{spn}(T)) = \mathcal{Z} \left( \mathbf{R}(p^3n, \operatorname{spn}(S)) + \mathbf{R}^\sharp(pn, \operatorname{spn}(\lambda_p(S))) \right). \quad (5.2.6)$$

If  $(\Gamma_{p,1}(S)^{\frac{1}{p}}, \Gamma_{p,2}(S)^{\frac{1}{p}}) \simeq (T_1, T_2)$ , then

$$(\Gamma_{p,1}(\lambda_p(S))^{\frac{1}{p}}, \Gamma_{p,2}(\lambda_p(S))^{\frac{1}{p}}) \simeq (\lambda_p(T_1), \lambda_p(T_2)).$$

Hence we have

$$\mathbf{R}^\sharp(pn, \operatorname{spn}(\lambda_p(S))) = \mathcal{N}^t \cdot \mathbf{R}^\sharp(n, \operatorname{spn}(\lambda_p(T))) - \mathbf{R}^\sharp(n, \operatorname{spn}(\lambda_p^2(S))), \quad (5.2.7)$$

that is,

$$\mathbf{R}^\sharp(n, \operatorname{spn}(\lambda_p(T))) = \mathcal{Z}(\mathbf{R}^\sharp(pn, \operatorname{spn}(\lambda_p(S))) + \mathbf{R}^\sharp(n, \operatorname{spn}(\lambda_p^2(S))). \quad (5.2.8)$$



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By Proposition 5.1.2, we also have

$$\begin{aligned} \mathbf{R}(p^2n, \text{spn}(T)) + 2p \mathbf{R}(n, \text{spn}(T)) \\ = \mathcal{M} \cdot \mathbf{R}(pn, \text{spn}(S)) + \mathbf{R}^\sharp(n, \text{spn}(\lambda_p(T))). \end{aligned} \quad (5.2.9)$$

If  $n$  is not divisible by  $p$ , then (5.2.4) comes directly from (5.2.5). Assume that  $n$  is divisible by  $p$ . Since  $\lambda_p^3(S) \simeq \lambda_p(S)$ , we have

$$\mathbf{R}^\sharp\left(\frac{n}{p}, \text{spn}(\lambda_p(S))\right) = \mathbf{R}^\sharp(n, \text{spn}(\lambda_p^2(S))). \quad (5.2.10)$$

Therefore, the theorem follows from equations (5.2.5), (5.2.6), (5.2.8) and (5.2.9).

If we replace  $\mathcal{N}$  by  $\tilde{\mathcal{N}}$ , then the proof of the case when  $\mathfrak{G}_{L,p}(2)$  is of  $E$ -type is quite similar to this.  $\square$

**Example 5.2.4.** Let  $p = 3$  and let  $L = \langle 1, 1, 2 \rangle$ . Then  $T = \langle 1, 2, 9 \rangle \in \mathcal{G}_{L,p}(2)$  and  $S_1 = \langle 1, 2, 27 \rangle \in \mathcal{G}_{L,p}(3)$ . In fact, the graph  $\mathfrak{G}_{L,p}(2)$  is of  $O$ -type and

$$\mathcal{G}_{L,p}(3) = \left\{ S_1, S_2 = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 6 \end{pmatrix}, S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 1 & 11 \end{pmatrix}, S_4 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 1 & 7 \end{pmatrix} \right\},$$

up to isometry. In this case, one may easily check that there are no rational numbers  $a_i$  and  $b_i$  satisfying the equation

$$r(n, T) = \sum_{i=1}^4 a_i \cdot r(3n, S_i) + \sum_{i=1}^4 b_i \cdot r(27n, S_i) \quad \text{for any integer } n.$$

Finally, assume that  $m \geq 3$ . Let  $T \in \mathcal{G}_{L,p}(m)$  and  $S \in \mathcal{G}_{L,p}(m+1)$  be  $\mathbb{Z}$ -lattices such that  $r(T^p, S) \neq 0$ . We additionally assume that  $\mathfrak{G}_{L,p}(m)$  is of  $O$ -type. Recall that  $\mathcal{M} = \left( \frac{r(T_i^p, S_j)}{o(T_i)} \right)$  and  $\mathcal{N} = \left( \frac{r(T_i^p, S_j)}{o(S_j)} \right)$ . We define  $\mathcal{Z} = (\mathcal{N}\mathcal{N}^t)^{-1}\mathcal{N}$ .

**Theorem 5.2.5.** Under the assumptions given above, if  $n$  is not divisible by

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$p$ , then

$$\mathbf{R}(n, \text{spn}(T)) = \mathcal{Z}(\mathbf{R}(pn, \text{spn}(S))) \quad \text{and} \quad \mathbf{R}(pn, \text{spn}(T)) = \mathcal{M} \cdot \mathbf{R}(n, \text{spn}(S)).$$

For an arbitrary integer  $n$ , we have

$$\begin{aligned} & p\mathbf{R}(p^2n, \text{spn}(T)) - p^2\mathbf{R}(n, \text{spn}(T)) \\ &= \mathcal{Z}\left(2p\mathbf{R}(p^3n, \text{spn}(S)) + p^2\mathbf{R}(pn, \text{spn}(S)) + \mathbf{R}^b(pn, \text{spn}(S))\right) - p\mathcal{M} \cdot \mathbf{R}(pn, \text{spn}(S)), \end{aligned}$$

where

$$\mathbf{R}^b(pn, \text{spn}(S)) = \left( \frac{o(\lambda_p(S_1))}{o(S_1)} r(pn, \text{gen}_p^{\lambda_p(S_1)}(S_1)), \dots, \frac{o(\lambda_p(S_v))}{o(S_v)} r(pn, \text{gen}_p^{\lambda_p(S_v)}(S_v)) \right)^t.$$

*Proof.* By Propositions 5.1.1 and 5.1.2, we have

$$\mathbf{R}(pn, \text{spn}(S)) = \mathcal{N}^t \cdot \mathbf{R}(n, \text{spn}(T)) - \mathbf{R}^\sharp\left(\frac{n}{p}, \text{spn}(\lambda_p(S))\right), \quad (5.2.11)$$

and

$$\begin{aligned} \mathbf{R}(pn, \text{spn}(T)) &= \mathcal{M} \cdot \mathbf{R}(n, \text{spn}(S)) \\ &\quad + \mathbf{R}^\sharp\left(\frac{n}{p}, \text{spn}(\lambda_p(T))\right) - 2p \cdot \mathbf{R}\left(\frac{n}{p}, \text{spn}(T)\right). \end{aligned} \quad (5.2.12)$$

The first two equations follow directly from (5.2.11) and (5.2.12).

Now by applying  $\lambda_p$ -transformation to the equation (5.2.11), we also have

$$\mathbf{R}^\sharp(pn, \text{spn}(\lambda_p(S))) = \mathcal{N}^t \cdot \mathbf{R}^\sharp(n, \text{spn}(\lambda_p(T))) - \mathbf{R}^\sharp\left(\frac{n}{p}, \text{spn}(\lambda_p^2(S))\right). \quad (5.2.13)$$

Our final ingredient is the following equation which is directly obtained from Proposition 3.2.9:

$$\begin{aligned} & p\mathbf{R}^\sharp(pn, \text{spn}(\lambda_p(S))) + p^2\mathbf{R}^\sharp\left(\frac{n}{p}, \text{spn}(\lambda_p(S))\right) - p\mathbf{R}^\sharp\left(\frac{n}{p}, \text{spn}(\lambda_p^2(S))\right) \\ &= \mathbf{R}^b(pn, \text{spn}(S)). \end{aligned} \quad (5.2.14)$$

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By multiplying  $\mathcal{Z}$  to (5.2.11), we have

$$\mathbf{R}(n, \text{spn}(T)) = \mathcal{Z} \left( \mathbf{R}(pn, \text{spn}(S)) + \mathbf{R}^\sharp \left( \frac{n}{p}, \text{spn}(\lambda_p(S)) \right) \right).$$

Hence we have

$$\begin{aligned} 2p\mathbf{R}(p^2n, \text{spn}(T)) + p^2\mathbf{R}(n, \text{spn}(T)) &= 2p\mathcal{Z} \left( \mathbf{R}(p^3n, \text{spn}(S)) + \mathbf{R}^\sharp(pn, \text{spn}(\lambda_p(S))) \right) \\ &\quad + p^2\mathcal{Z} \left( \mathbf{R}(pn, \text{spn}(S)) + \mathbf{R}^\sharp \left( \frac{n}{p}, \text{spn}(\lambda_p(S)) \right) \right). \end{aligned}$$

On the other hand, by combining (5.2.12) and (5.2.13), we have

$$\begin{aligned} \mathbf{R}(p^2n, \text{spn}(T)) + 2p\mathbf{R}(n, \text{spn}(T)) - \mathcal{M} \cdot \mathbf{R}(pn, \text{spn}(S)) \\ = \mathcal{Z} \left( \mathbf{R}^\sharp(pn, \text{spn}(\lambda_p(S))) + \mathbf{R}^\sharp \left( \frac{n}{p}, \text{spn}(\lambda_p^2(S)) \right) \right). \end{aligned}$$

The theorem follows from the above two equations and (5.2.14).  $\square$

## Chapter 6

# Genus-correspondences and representations of ternary quadratic forms

In this chapter, we consider genus-correspondences that respect spinor genus. We disprove the conjecture that every genus-correspondence between two ternary quadratic forms respects spinor genus, which was given Jagy in [12]. We modify this conjecture and prove this modified version. As samples of genus-correspondences, we define a reduced genus  $\text{Rgen}(N)$  for any ternary  $\mathbb{Z}$ -lattice  $N$ , and we construct genus-correspondences satisfying some additional properties by using the reduced genera.

Finally, we construct infinite family of genera of ternary  $\mathbb{Z}$ -lattices that possess (absolutely) complete systems of spinor exceptional integers.

### 6.1 Genus-correspondences

Let  $M$  be a ternary  $\mathbb{Z}$ -lattice on the quadratic space  $V$  and let  $N$  be a ternary  $\mathbb{Z}$ -lattice on  $V^n$  such that  $dN = n \cdot dM$  for some positive integer  $n$ . For an  $M' \in \text{gen}(M)$  and an  $N' \in \text{gen}(N)$ , if there is a representation  $\phi : (M')^n \rightarrow N'$ , then  $[N' : \phi((M')^n)] = n$ . This implies that  $nN' \subseteq \phi((M')^n)$  and hence  $(N')^n$  is represented by  $M'$ . The pair  $(N', M') \in \text{gen}(N) \times \text{gen}(M)$  satisfying the above property is called *a representable pair by scaling  $n$* , or

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simply *representable pair*. As pointed out by Jagy in [12], Chan proved that if  $(N, M)$  is a representable pair by scaling  $n$ , then for any  $M' \in \text{gen}(M)$ , there is a ternary  $\mathbb{Z}$ -lattice  $N' \in \text{gen}(N)$  such that  $(N', M')$  is a representable pair by scaling  $n$ , and conversely for any  $N'' \in \text{gen}(N)$ , there is a ternary  $\mathbb{Z}$ -lattice  $M'' \in \text{gen}(M)$  such that  $(N'', M'')$  is a representable pair by scaling  $n$ .

Let  $n$  be a positive integer. Let  $N$  and  $M$  be ternary  $\mathbb{Z}$ -lattices such that  $(N, M)$  is a representable pair by scaling  $n$ . A subset  $\mathfrak{S} \subseteq \text{gen}(N) \times \text{gen}(M)$  is called a *genus-correspondence* if it satisfies following conditions:

- (i) each element of  $\mathfrak{S}$  is a representable pair by scaling  $n$ ;
- (ii) for any  $N' \in \text{gen}(N)$ , there is a lattice  $M' \in \text{gen}(M)$  such that the pair  $(N', M')$  is contained in  $\mathfrak{S}$ ;
- (iii) for any  $M'' \in \text{gen}(M)$ , there is a lattice  $N'' \in \text{gen}(N)$  such that the pair  $(N'', M'')$  is contained in  $\mathfrak{S}$ .

Furthermore, if the genus-correspondence  $\mathfrak{S}$  satisfies the additional condition:

$$N'' \in \text{spn}(N') \iff M'' \in \text{spn}(M'),$$

for any  $(N', M'), (N'', M'') \in \mathfrak{S}$ , then  $\mathfrak{S}$  is called a *genus-correspondence respecting spinor genus*.

The following conjecture was given by Jagy in [12]:

**Conjecture.** *Given two genera  $G_1, G_2$  of positive ternary forms, with integral squarefree discriminant ratio and with a genus-correspondence, suppose that  $G_1, G_2$  have exactly the same number of spinor genera. Then the genus-correspondence respects spinor genus.*

The following example shows that the above conjecture is not true.

**Example 6.1.1.** Let  $N_1 = \langle 12 \rangle \perp \begin{pmatrix} 15 & 5 \\ 5 & 135 \end{pmatrix}$  and  $M_1 = \langle 1, 20, 80 \rangle$ . Then one may easily check that  $g^+(M_1) = g^+(N_1) = 2$ ,  $dN_1 = 15 \cdot M_1$ , and  $M_1^{15}$  is represented by  $N_1$ . The genus of  $N_1$  consists of, up to isometry, the following

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12 lattices:

$$\begin{aligned} N_1 &= \langle 12, 15, 135, 5, 0, 0 \rangle, & N_2 &= \langle 3, 7, 1200, 0, 0, 1 \rangle, & N_3 &= \langle 3, 60, 140, 20, 0, 0 \rangle \\ N_4 &= \langle 3, 27, 300, 0, 0, 1 \rangle, & N_5 &= \langle 27, 27, 40, 10, 10, 3 \rangle, & N_6 &= \langle 12, 28, 83, 12, 4, -4 \rangle, \\ N_7 &= \langle 12, 28, 75, 0, 0, 4 \rangle, & N_8 &= \langle 15, 35, 48, 0, 0, 5 \rangle, & N_9 &= \langle 7, 12, 300, 0, 0, 2 \rangle, \\ N_{10} &= \langle 12, 43, 60, 20, 0, 6 \rangle, & N_{11} &= \langle 8, 12, 303, 4, -2, 4 \rangle, & N_{12} &= \langle 12, 35, 60, 10, 0, 0 \rangle. \end{aligned}$$

Note that

$$N_1, N_2, N_3, N_4, N_5, N_6 \in \text{spn}(N_1) \text{ and } N_7, N_8, N_9, N_{10}, N_{11}, N_{12} \in \text{spn}(N_7).$$

The genus of  $M_1$  consists of, up to isometry, the following 6 lattices:

$$\begin{aligned} M_1 &= \langle 1, 20, 80, 0, 0, 0 \rangle, & M_2 &= \langle 5, 16, 20, 0, 0, 0 \rangle, & M_3 &= \langle 4, 20, 25, 10, 0, 0 \rangle, \\ M_4 &= \langle 4, 5, 80, 0, 0, 0 \rangle, & M_5 &= \langle 9, 9, 20, 0, 0, 1 \rangle, & M_6 &= \langle 4, 20, 21, 0, 2, 0 \rangle. \end{aligned}$$

Note that

$$M_1, M_2, M_3 \in \text{spn}(M_1) \text{ and } M_4, M_5, M_6 \in \text{spn}(M_4).$$

Define a genus-correspondence  $\mathfrak{S}$  as follows:

$$\begin{aligned} \mathfrak{S} &= \{(N_1, M_1), (N_9, M_1), (N_3, M_2), (N_7, M_2), (N_5, M_3), (N_{11}, M_3), \\ &\quad (N_2, M_4), (N_8, M_4), (N_6, M_5), (N_{10}, M_5), (N_4, M_6), (N_{12}, M_6)\}. \end{aligned}$$

Then one may easily check that the genus-correspondence  $\mathfrak{S}$  does not respect spinor genus.

We will prove that if  $(N, M)$  is a representable pair and  $g^+(N) = g^+(M)$ , then there is a genus-correspondence respecting spinor genus.

**Lemma 6.1.2.** *For ternary  $\mathbb{Z}$ -lattices  $N$  and  $M$ , assume that  $(N, M)$  is a representable pair by scaling  $n$ . Then for any  $N' \in \text{spn}(N)$ , there is a ternary  $\mathbb{Z}$ -lattice  $M' \in \text{spn}(M)$  such that  $(N', M')$  is a representable pair. Conversely, for any  $M'' \in \text{spn}(M)$  there is a  $\mathbb{Z}$ -lattice  $N'' \in \text{spn}(N)$  such that  $(N'', M'')$  is a representable pair.*

*Proof.* Since  $(N, M)$  is a representable pair, there is an isometry  $\sigma \in O(V)$  such that  $\sigma(M^n) \subseteq N$ . Let  $N' \in \text{spn}(N)$ . Then there are  $\sigma' \in O(V)$  and

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$\Sigma \in J_V$  such that  $N' = \sigma' \Sigma N$ . Note that

$$\sigma' \Sigma \sigma(M^n) \subseteq \sigma' \Sigma N = N'.$$

Since  $\sigma' \Sigma \sigma(M) = \sigma' \sigma(\sigma^{-1} \Sigma \sigma)(M) \in \text{spn}(M)$ , the lemma can be proved if we take  $M' = \sigma' \Sigma \sigma(M)$ . The proof of the converse is quite similar to this.  $\square$

For a representable pair  $(N, M)$ , define a bipartite graph  $\mathfrak{G}(N, M)$  as follows:

- the set of all vertices of the graph  $\mathfrak{G}(N, M)$  is  $\text{gen}(N)_S \cup \text{gen}(M)_S$ ;
- the bipartition of the graph  $\mathfrak{G}(N, M)$  is given by  $(\text{gen}(N)_S, \text{gen}(M)_S)$ ;
- two vertices  $\text{spn}(N') \in \text{gen}(N)_S$  and  $\text{spn}(M') \in \text{gen}(M)_S$  of the graph  $\mathfrak{G}(N, M)$  are contained in an edge if and only if there exist lattices  $N'' \in \text{spn}(N')$  and  $M'' \in \text{spn}(M')$  such that  $(N'', M'')$  is a representable pair.

For a vertex  $\text{spn}(N') \in \text{gen}(N)_S$  of the graph  $\mathfrak{G}(N, M)$ , we define

$$\mathcal{N}(\text{spn}(N')) = \{\text{spn}(M') \in \text{gen}(M)_S \mid \text{spn}(M') \text{ adjacent to } \text{spn}(N')\}.$$

For a vertex  $\text{spn}(M') \in \text{gen}(M)_S$ , the set  $\mathcal{N}(\text{spn}(M'))$  is defined similarly.

**Lemma 6.1.3.** *For ternary  $\mathbb{Z}$ -lattices  $N$  and  $M$ , assume  $g^+(N) = g^+(M)$  and  $(N, M)$  is a representable pair. Then the graph  $\mathfrak{G}(N, M)$  is a regular bipartite graph.*

*Proof.* For vertices  $\text{spn}(N') \in \text{gen}(N)_S$  and  $\text{spn}(M') \in \text{gen}(M)_S$  of the graph  $\mathfrak{G}(N, M)$ , assume that  $\text{spn}(M') \in \mathcal{N}(\text{spn}(N'))$ . By Lemma 6.1.2, we may assume that  $(N', M')$  is a representable pair. Furthermore we may assume that  $(M')^n$  is a sublattice of  $N'$ . Let  $\text{spn}(N'')$  be another vertex in  $\text{gen}(N)_S$ . There is a  $\Sigma \in J_V$  such that  $N'' = \Sigma N'$ . Then clearly,  $(\Sigma N', \Sigma M')$  is a representable pair, so  $\text{spn}(\Sigma M') \in \mathcal{N}(\text{spn}(N''))$ . Note that for any two lattices  $M', M'' \in \text{gen}(M)$ ,

$$M' \in \text{spn}(M'') \quad \text{if and only if} \quad \Sigma M' \in \text{spn}(\Sigma M'').$$

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Therefore

$$|\mathcal{N}(\text{spn}(N'))| = |\mathcal{N}(\text{spn}(N''))|.$$

Hence, we know that there are same number of vertices in  $\mathcal{N}(\text{spn}(N'))$ , for any  $\text{spn}(N') \in \text{gen}(N)_S$ . The lemma follows directly from the fact that  $g^+(N) = g^+(M)$ .  $\square$

A *matching* of a graph is a set of edges without common vertices. A *perfect matching* of a graph is a matching in which every vertex of the graph is incident to exactly one edge of the matching. The following theorem known as **Hall's Marriage Theorem** is quite useful in our situation:

**Theorem.** *Suppose that  $G$  is a bipartite graph with bipartition  $(A, B)$ . Then  $G$  has a perfect matching if and only if  $|A| = |B|$  and for any  $S \subseteq V(G)$ ,  $|N_G(S)| \geq |S|$ , where  $V(G)$  is the set of all vertices of  $G$  and  $N_G(S)$  is the set of all vertices adjacent to  $v$  for some  $v \in S$ .*

As a corollary of this theorem, it is well known that any regular bipartite graph has a perfect matching.

**Theorem 6.1.4.** *Let  $N$  and  $M$  be ternary  $\mathbb{Z}$ -lattices such that  $(N, M)$  is a representable pair and  $g^+(N) = g^+(M)$ . Then there is a genus-correspondence respecting spinor genus.*

*Proof.* Since the graph  $\mathfrak{G}(N, M)$  is a regular bipartite graph by Lemma 6.1.3, there is a perfect matching, say  $P$ , in the graph  $\mathfrak{G}(N, M)$ . We define a genus-correspondence  $\mathfrak{S}$  as follows: for a pair  $(N', M')$  in  $\text{gen}(N) \times \text{gen}(M)$ ,  $(N', M') \in \mathfrak{S}$  if and only if  $(N', M')$  is a representable pair and two vertices  $\text{spn}(N')$ ,  $\text{spn}(M')$  in  $\mathfrak{G}(N, M)$  are contained in an edge which is in the perfect matching  $P$ . Now the genus-correspondence  $\mathfrak{S}$  respects spinor genus by Lemma 6.1.2.  $\square$

**Example 6.1.5.** *In Example 6.1.1, define a genus-correspondence  $\mathfrak{S}'$  as follows:*

$$\begin{aligned} \mathfrak{S}' = \{ & (N_1, M_1), (N_2, M_1), (N_3, M_2), (M_4, M_2), (N_5, M_3), (N_6, M_3) \\ & (N_7, M_4), (N_8, M_4), (N_9, M_5), (N_{10}, M_5), (N_{11}, M_6), (N_{12}, M_6) \}. \end{aligned}$$

*One may easily check that the genus-correspondence  $\mathfrak{S}'$  respects spinor genus.*



## 6.2 Reduced genera

Let  $N$  be a primitive ternary  $\mathbb{Z}$ -lattice. In this section, we define a reduced genus  $\text{Rgen}(N)$  determined by  $N$  such that there is an  $N' \in \text{Rgen}(N)$  such that  $(N, N')$  is a representable pair by scaling  $n$ , for some positive integer  $n$ . We also define a subset  $\mathfrak{R} = \mathfrak{R}(N) \subseteq \text{gen}(N) \times \text{Rgen}(N)$  which is, in fact, a genus-correspondence respecting spinor genus.

Recall that a non-classic ternary  $\mathbb{Z}$ -lattice  $N$  is called an  $H$ -type at  $p$  if  $g^+(N) = g^+(\lambda_p(N))$ . We say  $N$  is of  $O$ -type ( $E$ -type) at  $p$  if  $N \in \mathcal{G}_{L,p}(m)$  and the graph  $\mathfrak{G}_{L,p}(m)$  is of  $O$ -type ( $E$ -type, respectively), where the definitions of  $\mathcal{G}_{L,p}(m)$  and  $\mathfrak{G}_{L,p}(m)$  can be found in Chapter 4.

**Definition 6.2.1.** Let  $N$  be a primitive ternary  $\mathbb{Z}$ -lattice. A prime  $p$  dividing  $4 \cdot dN$  is called a *removable prime* for  $\text{gen}(N)$  if one of the following conditions hold:

- (i) if  $\text{ord}_p(4 \cdot dN) \geq 2$ , then  $N$  is of  $H$ -type at  $p$  and is not of  $E$ -type at  $p$ ;
- (ii) if  $\text{ord}_p(4 \cdot dN) = 1$ , then  $N \in \mathcal{G}_{L,p}(1)$  and the graph  $\mathcal{G}_{L,p}(0)$  is of  $O$ -type.

If there is no removable prime  $p$  dividing  $4 \cdot dN$ , then we say that  $\text{gen}(N)$  is a *reduced genus*.

Let  $N$  be a primitive ternary  $\mathbb{Z}$ -lattice and let  $p$  be a removable prime for  $\text{gen}(N)$ . A  $\mathbb{Z}$ -lattice  $M$  is called a  $p$ -*descendent* of  $N$  if

$$\begin{cases} M \simeq \lambda_p(N) & \text{if a prime } p \text{ satisfies the condition (i),} \\ M \simeq \Gamma_{p,1}(N)^{\frac{1}{p}} \text{ or } \Gamma_{p,2}(N)^{\frac{1}{p}} & \text{if a prime } p \text{ satisfies the condition (ii),} \end{cases}$$

Furthermore, a  $\mathbb{Z}$ -lattice  $M$  is called a *descendent* of  $N = N_0$  if there are primes  $p_i$  and  $\mathbb{Z}$ -lattices  $N_i$  for  $i = 1, 2, \dots, t$  such that for each  $i$ ,  $N_i$  is a  $p_i$ -descendent of  $N_{i-1}$  and  $N_t = M$ .

Let  $M$  be a descendent of  $N$  such that there is no removable prime for  $\text{gen}(M)$ . Assume that  $M'$  is another descendent of  $N$  such that there is no removable prime for  $\text{gen}(M')$ . Then one may easily show that  $M' \in \text{spn}(M)$ . In general, for any  $N' \in \text{spn}(N)$  and any descendent  $M''$  of  $N'$  such that there is no removable prime for  $\text{gen}(M'')$ ,  $M'' \in \text{spn}(M)$ . This comes from

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the fact that  $\lambda_p(N') \in \text{spn}(\lambda_p(N))$  for any  $N' \in \text{spn}(N)$  and if  $p$  satisfies the condition (ii), then by Lemmas 4.2.9 and 4.2.10,

$$\text{spn}(\Gamma_{p,1}(N)^{\frac{1}{p}}) = \text{spn}(\Gamma_{p,2}(N)^{\frac{1}{p}}). \quad (6.2.1)$$

From now on, the *reduced* genus  $\text{Rgen}(N)$  of  $N$  is defined by  $\text{gen}(M)$ . Note that there are same number of spinor genera in  $\text{gen}(N)$  and  $\text{Rgen}(N)$ , however, in general, the class number of  $\text{Rgen}(N)$  is less than the class number of  $\text{gen}(N)$ .

Define a subset  $\mathfrak{R} = \mathfrak{R}(N) \subseteq \text{gen}(N) \times \text{Rgen}(N)$  as follows: for a pair  $(N', M')$  in  $\text{gen}(N) \times \text{Rgen}(N)$ ,  $(N', M') \in \mathfrak{R}$  if and only if  $M'$  is a descendent of  $N'$ . Note that for any  $N' \in \text{gen}(N)$ , there is a lattice  $M' \in \text{Rgen}(N)$  such that  $M'$  is a descendent of  $N'$  and conversely for any  $M'' \in \text{Rgen}(N)$ , there is a lattice  $N'' \in \text{gen}(N)$  such that  $M''$  is a descendent of  $N''$ . Furthermore one may easily show that for pairs  $(N', M'), (M'', N'') \in \mathfrak{R}$ ,

$$N' \in \text{spn}(N'') \text{ if and only if } M' \in \text{spn}(M''), \quad (6.2.2)$$

by Remark 3.1.4 and Lemmas 4.2.9, 4.2.10.

Let  $M$  be a descendent of  $N = N_0$  in  $\text{Rgen}(N)$ . Then, there are primes  $p_i$  and  $\mathbb{Z}$ -lattices  $N_i$  for  $i = 1, 2, \dots, t$  such that for each  $i$ ,  $N_i$  is a  $p_i$ -descendent of  $N_{i-1}$  and  $N_t = M$ . Define an integer  $\delta(i)$  for  $1 \leq i \leq t-1$  as follows:

- (i) if  $N_i \simeq \lambda_p(N_{i-1})$ , then  $\delta(i) = p$  or  $p^2$ , which satisfies the equation

$$\Lambda_p(N_{i-1})^{\frac{1}{\delta(i)}} = \lambda_p(N_{i-1});$$

- (ii) if  $N_i \simeq \Gamma_{p,1}(N_{i-1})^{\frac{1}{p}}$  or  $\Gamma_{p,2}(N_{i-1})^{\frac{1}{p}}$ , then  $\delta(i) = p$ .

Furthermore, define an integer

$$n = n(N) = \prod_{1 \leq i \leq t} \delta(i). \quad (6.2.3)$$

Now, assume that  $(N', M') \in \mathfrak{R}$ . Then one may easily check that  $(M')^n$  is represented by  $N'$ . Furthermore, if  $dN' = n \cdot dM'$ , then the condition (6.2.2)

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implies that the subset  $\mathfrak{R}$  is a genus-correspondence respecting spinor genus. In fact,  $\mathfrak{R}$  satisfies the following additional property:

**Lemma 6.2.2.** *Let  $N$  be a primitive ternary  $\mathbb{Z}$ -lattice and let  $(N', M')$  be a pair contained in  $\mathfrak{R} = \mathfrak{R}(N)$ . For any integer  $k$ , there exists a lattice  $R(N', k) \in \text{spn}(M')$  such that  $k$  is represented by  $R(N', k)$  if and only if  $nk$  is represented by  $N'$ , where  $n = n(N)$  is the integer defined in the equation (6.2.3).*

*Proof.* Since  $M'$  is a descendent of  $N' = N'_0$ , there are primes  $p_i$  and ternary  $\mathbb{Z}$ -lattices  $N'_i$  for  $i = 1, 2, \dots, t$  such that for each  $i$ ,  $N'_i$  is a  $p_i$ -descendent of  $N'_{i-1}$  and  $N'_t = M'$ .

Suppose that  $N'_i = \Gamma_{p,1}(N'_{i-1})^{\frac{1}{p_i}}$  or  $\Gamma_{p,2}(N'_{i-1})^{\frac{1}{p_i}}$ , for some  $1 \leq i \leq t$ . Note that  $\delta(1) \cdots \delta(i)k$  is represented by  $N'_{i-1}$  if and only if  $\delta(1) \cdots \delta(i-1)k$  is represented by  $\Gamma_{p,1}(N'_{i-1})^{\frac{1}{p_i}}$  or  $\Gamma_{p,2}(N'_{i-1})^{\frac{1}{p_i}}$ . Since  $\mathfrak{G}_{L,p}(0)$  is of  $O$ -type, we know that  $\text{spn}(\Gamma_{p,1}(N'_{i-1})^{\frac{1}{p_i}}) = \text{spn}(\Gamma_{p,2}(N'_{i-1})^{\frac{1}{p_i}})$ .

Assume that  $N'_i = \lambda_{p_i}(N'_{i-1})$  for  $1 \leq i \leq t$ . Furthermore, assume that the  $\frac{1}{2}\mathbb{Z}_p$ -modular component in a Jordan splitting of  $(N'_{i-1})_{p_i}$  is non-zero isotropic. Since  $N'_{i-1}$  is not of  $E$ -type at  $p_i$ , we have

$$\{\Gamma_{p_i,v}(\Gamma_{p_i,u}(N'_{i-1})^{\frac{1}{p_i}}) \mid u, v = 1 \text{ or } 2\} \subseteq \text{spn}(\lambda_{p_i}(N'_{i-1})),$$

Similarly as above, one may easily show that  $\delta(1) \cdots \delta(i)k$  is represented by  $N'_{i-1}$  if and only if  $\delta(1) \cdots \delta(i-1)k$  is represented by one of the lattices in

$$\{\Gamma_{p_i,v}(\Gamma_{p_i,u}(N'_{i-1})^{\frac{1}{p_i}}) \mid u, v = 1 \text{ or } 2\}.$$

If the  $\frac{1}{2}\mathbb{Z}_p$ -modular component is zero or non-zero anisotropic in a Jordan splitting of  $(N'_{i-1})_{p_i}$ , then one may easily show that  $\delta(1) \cdots \delta(i)k$  is represented by  $N'_{i-1}$  if and only if  $\delta(1) \cdots \delta(i-1)k$  is represented by  $\lambda_{p_i}(N'_{i-1})$ . This completes the proof.  $\square$

**Corollary 6.2.3.** *Under the same notations given above, if  $k$  is represented by all lattices in  $\text{spn}(M')$ , then  $nk$  is represented by all lattices in  $\text{spn}(N')$ .*

*Proof.* The corollary follows directly from the above lemma.  $\square$

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The following example was first introduced by Jagy in [12] under the assumption that  $n$  is squarefree.

**Example 6.2.4.** *Let  $n$  be an integer that is represented by a sum of two integral squares. Define a ternary  $\mathbb{Z}$ -lattice  $N = \langle 1, 1, 16n \rangle$ . Note that for every prime  $p$  satisfying  $p \equiv 3 \pmod{4}$ , the value  $\text{ord}_p(n)$  is even. Then one may easily check that  $\langle 1, 1, 16 \rangle$  is a descendent of  $N$  and*

$$R\text{gen}(N) = \text{gen}(\langle 1, 1, 16 \rangle).$$

*Note that  $g^+(\langle 1, 1, 16 \rangle) = 2$  and the genus of  $\langle 1, 1, 16 \rangle$  consists of, up to isometry, following two lattices:*

$$\langle 1, 1, 16 \rangle \text{ and } \langle 2, 2, 5, 1, 1, 0 \rangle.$$

*Let  $\tilde{N} \in \text{gen}(N)$  be a  $\mathbb{Z}$ -lattice whose descendent is  $\langle 2, 2, 5, 1, 1, 0 \rangle$ . Since  $g^+(N) = 2$ , we know that*

$$\text{gen}(N) = \{\text{spn}(\langle 1, 1, 16n \rangle), \text{spn}(\tilde{N})\}.$$

*Since 1 is represented by  $\langle 1, 1, 16 \rangle$  and is not represented by  $\langle 2, 2, 5, 1, 1, 0 \rangle$ , all lattices in  $\text{spn}(\langle 1, 1, 16n \rangle)$  represent  $n$ , and all lattices in  $\text{spn}(\tilde{N})$  do not represent  $n$  by Lemma 6.2.2. Hence, for  $\mathbb{Z}$ -lattices  $N', N'' \in \text{gen}(N)$ ,  $N'$  and  $N''$  are spinor equivalent if and only if  $N'$  and  $N''$  represent  $n$  simultaneously, or not.*

### 6.3 Spinor character theory

In this section, we investigate the property explained in Example 6.2.4 in detail. We show that in some particular case, spinor equivalences on a genus can be completely determined by representations of spinor exceptional integers.

Let  $N$  be a non-classic ternary  $\mathbb{Z}$ -lattice  $N$  and let  $c$  be an integer that is represented by the genus of  $N$ . Then we know either every spinor genus in the genus represents  $c$  (that is, there is a lattice in each spinor genus representing

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$c$ ) or else precisely half of all the spinor genera represent  $c$ . In the latter case,  $c$  is called a *spinor exceptional integer* for  $\text{gen}(N)$  and the half-genus that represents (doesn't represent)  $c$  is called *good* (*bad*, respectively) half-genus with respect to  $c$ . (for details, see [1]). The integer  $c$  is a spinor exceptional integer for  $\text{gen}(N)$  if and only if

$$\begin{aligned} c \neq 0, \quad -c \cdot dN \notin (\mathbb{Q}^\times)^2, \quad \theta(J_N) \subseteq N_{E/F}(J_E) \text{ and} \\ \theta(N_p, c) = N_p(E) \text{ for any finite prime } p, \end{aligned} \quad (6.3.1)$$

where  $E = \mathbb{Q}(\sqrt{-c \cdot dN})$  (see Theorem 2.3.3).

Assume that  $g^+(N) = 2^r$ . Suppose that for  $r$  spinor exceptional integers  $c_1, c_2, \dots, c_r$  of  $\text{gen}(N)$ , multi-quadratic extension

$$\mathbb{Q}(\sqrt{-c_1 dN}, \dots, \sqrt{-c_r dN}) / \mathbb{Q}$$

has degree  $2^r$ . Then the set  $\{c_1, \dots, c_r\}$  is called a *complete system of spinor exceptional integers* for  $\text{gen}(N)$ . Let  $\{c_1, \dots, c_r\}$  be a complete system of spinor exceptional integers for  $\text{gen}(N)$ . For a spinor genus  $\text{spn}(N')$  in  $\text{gen}(N)$ , we define  $\chi_i(\text{spn}(N')) = \pm 1$  with  $+1$  if and only if  $\text{spn}(N')$  represents  $c_i$  for  $1 \leq i \leq r$ . And define a map  $\chi : \text{gen}(N)_S \longrightarrow \{\pm 1\}^r$  given by

$$\chi(\text{spn}(N')) = (\chi_1(\text{spn}(N')), \dots, \chi_r(\text{spn}(N'))). \quad (6.3.2)$$

Then it is known that the map  $\chi$  is bijective (see [1]).

Now, we introduce the lemma proved by Hsia and Jöchner in [11]:

**Lemma.** *Let  $M$  be a positive integral  $\mathbb{Z}$ -lattice of rank three or more and of determinant  $d$ ,  $C$  a finite collection of primes such that for each  $p \in C$ ,  $M_p$  is isotropic. If  $t$  is a sufficiently large integer and divisible only by primes from  $C$ , then  $tN$  is representable by  $M$  for every lattice  $N$  which is representable by  $\text{spn}^+(M)$ .*

From the above lemma, we can prove the following lemma:

**Lemma 6.3.1.** *For a ternary  $\mathbb{Z}$ -lattice  $N$ , let  $c$  be a spinor exceptional integer for  $\text{gen}(N)$ . Then there is an integer  $t \in \mathbb{Z}$  such that  $ct^2$  is a spinor*

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*exceptional integer for  $\text{gen}(N)$  and  $ct^2$  is represented by all lattices belonging to the good half-genus with respect to  $ct^2$ .*

*Proof.* Take a prime  $p$  such that  $-c \cdot 4dN$  is square at  $p$  and  $(p, 2d) = 1$ . Let  $t$  be a some power of  $p$  that is sufficiently large. Since  $N_p$  is isotropic,  $ct^2$  is represented by all lattices in the good half-genus with respect to  $c$  by the above lemma. Show that  $ct^2$  is a spinor exceptional integer for  $\text{gen}(N)$ . By the equation (6.3.1), it is enough to show that  $\theta(N_q, ct^2) = N_q(E)$  for any finite prime  $q$ . Since  $-ct^2 \cdot 4dN$  is square at  $p$ ,  $\theta(N_p, ct^2) = N_p(E)$ . Assume  $q \neq p$ . Note that  $\text{ord}_q(c) = \text{ord}_q(ct^2)$ . Since  $\theta(N_q, c) = N_q(E)$ , we know that  $\theta(N_q, ct^2) = N_q(E)$  by Theorems 2.3.4, 2.3.5.  $\square$

Note that the good half-genus with respect to  $c$  coincides with the good half-genus with respect to  $ct^2$  in Lemma 6.3.1.

**Definition 6.3.2.** Let  $\{c_1, \dots, c_r\}$  be a complete system of spinor exceptional integers for a ternary  $\mathbb{Z}$ -lattice  $N$  with  $g^+(N) = 2^r$ . If each  $c_i$  is represented by all lattices in the good half-genus with respect to  $c_i$  for any  $1 \leq i \leq r$ , then we say  $\{c_1, \dots, c_r\}$  an *absolutely complete system of spinor exceptional integers* for  $\text{gen}(N)$ .

**Proposition 6.3.3.** *Let  $N$  be a ternary  $\mathbb{Z}$ -lattice with  $g^+(N) = 2^r$ . Suppose that there is a complete system of spinor exceptional integers  $\{c_1, \dots, c_r\}$  for  $\text{gen}(N)$ . There are integers  $t_1, \dots, t_r$  such that  $\{c_1 t_1^2, \dots, c_r t_r^2\}$  is an absolutely complete system of spinor exceptional integers for  $\text{gen}(N)$ . In particular, the good half-genus with respect to  $c_i$  coincides with the good half-genus with respect to  $c_i t_i^2$ , for any  $1 \leq i \leq r$ .*

*Proof.* The proposition follows directly from Lemma 6.3.1.  $\square$

Assume  $\{c_1, \dots, c_r\}$  is an absolutely complete system of spinor exceptional integers for  $\text{gen}(N)$ . For  $N' \in \text{gen}(N)$ , define  $\tilde{\chi}_i(N') = \pm 1$  for  $1 \leq i \leq r$  with  $+1$  if and only if  $N'$  represents  $c_i$ . And define a map  $\tilde{\chi} : \text{gen}(N) \rightarrow \{\pm 1\}^r$  given by

$$\tilde{\chi}(N') = (\tilde{\chi}_1(N'), \dots, \tilde{\chi}_r(N')). \quad (6.3.3)$$

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The following theorem shows that any spinor equivalences in  $\text{gen}(N)$  are completely determined by the map  $\tilde{\chi}$ .

**Theorem 6.3.4.** *Let  $N$  be a ternary  $\mathbb{Z}$ -lattice with  $g^+(N) = 2^r$  and let  $\{c_1, \dots, c_r\}$  be an absolutely complete system of spinor exceptional integers for  $\text{gen}(N)$ . Then for any two lattices  $N', N'' \in \text{gen}(N)$ ,  $N' \in \text{spn}(N'')$  if and only if*

$$\tilde{\chi}(N') = \tilde{\chi}(N''),$$

where  $\tilde{\chi}$  is the map from  $\text{gen}(N)$  to  $\{\pm 1\}^r$  defined above.

*Proof.* By the definition of  $\chi$  in (6.3.2),  $\text{spn}(N') = \text{spn}(N'')$  if and only if

$$\chi(\text{spn}(N')) = \chi(\text{spn}(N'')).$$

Equivalently,  $c_i$  represented by  $\text{spn}(N')$  if and only if  $c_i$  is represented by  $\text{spn}(N'')$  for any  $1 \leq i \leq r$ . Equivalently,  $c_i$  is represented by  $N'$  if and only if  $c_i$  is represented by  $N''$ , for any  $1 \leq i \leq r$ . Then by the definition of  $\tilde{\chi}$  in (6.3.3), it is equivalent to  $\tilde{\chi}(N') = \tilde{\chi}(N'')$ .  $\square$

Let  $N$  be a primitive ternary  $\mathbb{Z}$ -lattice and  $M$  be a descendent of  $N = N_0$  in  $\text{Rgen}(N)$ . Then, there are primes  $p_i$  and  $\mathbb{Z}$ -lattices  $N_i$  for  $i = 1, 2, \dots, t$  such that for each  $i$ ,  $N_i$  is a  $p_i$ -descendent of  $N_{i-1}$  and  $N_t = M$ . As given in Section 6.2, the subset  $\mathfrak{R} = \mathfrak{R}(N) \subseteq \text{gen}(N) \times \text{Rgen}(N)$  is defined as follows: for a pair  $(N', M')$  in  $\text{gen}(N) \times \text{Rgen}(N)$ ,  $(N', M') \in \mathfrak{R}$  if and only if  $M'$  is a descendent of  $N'$ .

**Lemma 6.3.5.** *Let  $N$  be a primitive ternary  $\mathbb{Z}$ -lattice. An integer  $c$  is a spinor exceptional integer for  $\text{Rgen}(N)$  if and only if  $nc$  is a spinor exceptional integer for  $\text{gen}(N)$ , where  $n = n(N)$  is an integer defined in the equation 6.2.3. In particular,  $\text{spn}(N')$  is contained in the good half-genus with respect to  $nc$  if and only if  $\text{spn}(M')$  is contained in the good half-genus with respect to  $c$ , for any  $(N', M') \in \mathfrak{R}$ .*

*Proof.* Let  $(N', M')$  be a pair in  $\mathfrak{R}$ . Assume that  $nc$  is represented by  $\text{spn}(N')$ . Without loss of generality, we may assume that  $N'$  represents  $nc$ . Then there is a lattice  $R(N', c) \in \text{spn}(M')$  that represents  $c$  by Lemma 6.2.2.

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Therefore  $c$  is represented by  $\text{spn}(M')$ . Conversely, Assume  $c$  is represented by  $\text{spn}(M')$ . we may assume that  $c$  is represented by a lattice  $M'$ . Then  $N'$  represents  $nc$  since  $(M')^n$  is represented by  $N'$ .  $\square$

**Lemma 6.3.6.** *Let  $N$  be a primitive ternary  $\mathbb{Z}$ -lattice with  $g^+(N) = 2^r$ . If  $\{c_1, \dots, c_r\}$  is a complete system of spinor exceptional integers for  $\text{Rgen}(N)$ , then  $\{nc_1, \dots, nc_r\}$  is a complete system of spinor exceptional integers for  $\text{gen}(N)$ , where  $n = n(N)$  is an integer defined in the equation (6.2.3).*

*Proof.* The proof follows directly from Lemma 6.3.5.  $\square$

The following theorem shows that we can find an absolutely complete system of spinor exceptional integers for  $\text{gen}(N)$  from that of  $\text{Rgen}(N)$  immediately.

**Theorem 6.3.7.** *Let  $N$  be a primitive ternary  $\mathbb{Z}$ -lattice with  $g^+(N) = 2^r$ . Suppose that there is an absolutely complete system of spinor exceptional integers  $\{c_1, \dots, c_r\}$  for  $\text{Rgen}(N)$ . Then  $\{nc_1, \dots, nc_r\}$  is an absolutely complete system of spinor exceptional integers for  $\text{gen}(N)$ , where  $n = n(N)$  is an integer defined in the equation (6.2.3).*

*Proof.* This follows directly from Corollary 6.2.3 and Lemma 6.3.6.  $\square$

**Example 6.3.8.** *Let  $n$  be an integer whose prime factor is congruent to 1 or 9 modulo 20. Define a  $\mathbb{Z}$ -lattice*

$$N = \langle 1, 20, 400n \rangle.$$

*Then one may easily show that  $\langle 1, 20, 400 \rangle$  is a descendent of  $N$ , and*

$$\text{Rgen}(N) = \text{gen}(\langle 1, 20, 400 \rangle).$$

*The genus of  $\langle 1, 20, 400 \rangle$  consists of, up to isometry, the following 12 lattices:*

$$\begin{aligned} M_1 &= \langle 1, 20, 400, 0, 0, 0 \rangle, & M_2 &= \langle 9, 9, 100, 0, 0, 1 \rangle, & M_3 &= \langle 4, 45, 45, 5, 0, 0 \rangle, \\ M_4 &= \langle 4, 5, 400, 0, 0, 0 \rangle, & M_5 &= \langle 16, 20, 29, 0, 8, 0 \rangle, & M_6 &= \langle 1, 80, 100, 0, 0, 0 \rangle, \\ M_7 &= \langle 4, 25, 80, 0, 0, 0 \rangle, & M_8 &= \langle 5, 16, 100, 0, 0, 0 \rangle, & M_9 &= \langle 4, 20, 101, 0, 2, 0 \rangle, \\ M_{10} &= \langle 16, 20, 25, 0, 0, 0 \rangle, & M_{11} &= \langle 4, 20, 105, 10, 0, 0 \rangle, & M_{12} &= \langle 4, 21, 100, 0, 0, 2 \rangle, \end{aligned}$$



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where  $g^+(\langle 1, 20, 400 \rangle) = 4$  and

$$\begin{aligned} M_1, M_2, M_3 &\in \text{spn}(M_1), & M_4, M_5, M_6 &\in \text{spn}(M_4), \\ M_7, M_8, M_9 &\in \text{spn}(M_7) & \text{and } M_{10}, M_{11}, M_{12} &\in \text{spn}(M_{10}). \end{aligned}$$

One may easily check that  $\{1, 5\}$  is a complete system of spinor exceptional integers for  $R\text{gen}(N)$ . Note that  $g^+(N) = 4$  and  $\{n, 5n\}$  is a complete system of spinor exceptional integers for  $\text{gen}(N)$  by Lemma 6.3.6. One may also check that  $\{1 \cdot 7^2, 5 \cdot 5^2\}$  is an absolutely complete system of spinor exceptional integers for  $R\text{gen}(N)$ . Therefore  $\{7^2n, 5^3n\}$  is an absolutely complete system of spinor exceptional integers for  $\text{gen}(N)$  by Theorem 6.3.7. For  $N' \in \text{gen}(N)$ , define  $\tilde{\chi}_1(N')$  ( $\tilde{\chi}_2(N')$ ) =  $\pm 1$  with  $+1$  if and only if  $N'$  represents  $7^2n$  ( $5^3n$ , respectively). And put

$$\tilde{\chi}(N') = (\tilde{\chi}_1(N'), \tilde{\chi}_2(N')).$$

Then for any lattices  $N', N'' \in \text{gen}(N)$ , we have

$$N' \in \text{spn}(N'') \text{ if and only if } \tilde{\chi}(N') = \tilde{\chi}(N''),$$

by Theorem 6.3.4.

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## 국문초록

이 논문에서는 (양의 정부호이고 정수계수인) 삼변수 이차형식에 의한 정수 표현의 개수 사이의 산술적 관계에 대하여 논의한다. 삼변수 이차형식  $f$  가 정수  $n$  을 표현하는 개수를  $r(n, f)$  라고 하고  $f$  가  $\mathbb{Z}_p$  위에서 0 이 아닌 해를 갖는 소수  $p$  를 고정하자. 우리는 적당한 조건하에서,  $f$  와  $p$  에 의해 결정되는 지너스  $\mathfrak{G}$  가 존재하여  $\mathfrak{G}$  에 포함되는 이차형식  $g$  들에 대한  $r(pn, g)$  와  $r(p^3n, g)$  그리고 정확히 계산가능한 나머지 항의 선형 합으로  $r(n, f)$  를 표현할 수 있음을 보인다.

이 논문에서는 또한 스피너 지너스를 보존하는 삼변수 이차형식 사이의 지너스-대응에 대하여 논의한다. 우리는 Jagy 의 추측이 틀렸음을 보이고 이를 수정하여 스피너 지너스를 보존하는 지너스-대응이 항상 존재함을 증명한다. 또한, 특별한 성질을 만족하는 지너스-대응을 건설하고 이를 이용하여 표현 여부로 스피너 지너스를 완벽히 분류할 수 있는 정수 집합이 존재하는 삼변수 이차형식의 지너스를 무한히 많이 건설한다.

**주요어휘:** 삼변수 이차형식의 표현, Watson 변환, 삼변수 이차형식의 그래프, 지너스-대응, 스피너 지너스를 분류하는 정수 집합

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